AS1056 - Mathematics for Actuarial Science. Chapter 7, Tutorial 2.

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The Riemann Sum

Let $f : [a, b] \to \mathbb{R}$ be a function defined on a closed interval $[a, b]$ of the real numbers, $\mathbb{R}$, and $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a, b]$, that is,

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

A **Riemann sum** S of f over $[a, b]$ with partition P is defined as

$$S(f, n) = \sum_{i=1}^{n} f(x_i^*) \, \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and $x_i^* \in [x_{i-1}, x_i]$.
It is often convenient to work with equal-sized partitions, i.e., $\Delta x_i = \frac{b-a}{n}, \ i = 1, ..., n.$

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- If $f(x_i^*) = \inf f([x_{i-1}, x_i])$ (i.e. the smallest f over $[x_{i-1}, x_i]$):

$$S_{\text{lower}}(f, n) = \sum_{i=1}^{n} \inf f([x_{i-1}, x_i])\, \Delta x_i \underset{\uparrow}{=} \frac{b-a}{n} \sum_{i=1}^{n} f_i^{\text{Low}}$$

under equal-sized partitions

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$\longrightarrow$ **Lower Riemann sum**

- If $f(x_i^*) = \sup f([x_{i-1}, x_i])$ (i.e. the largest f over $[x_{i-1}, x_i]$)

$$S_{\text{upper}}(f, n) = \sum_{i=1}^{n} \sup f([x_{i-1}, x_i]) \, \Delta x_i \underset{\uparrow}{=} \frac{b-a}{n} \sum_{i=1}^{n} f_i^{\text{High}}$$

under equal-sized partitions

$\longrightarrow$ **Upper Riemann sum**

The Riemann Integral

Definition

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, i.e. there is an $M \in R$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.
The function f is said to be **Riemann integrable** if its lower and upper integrals are the same, that is, if both lower and upper Riemann sums converge to the same value.
When this happens we define:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \underbrace{\lim_{n\to\infty} S_{\text{lower}}(f,n)}_{=\underline{\int_a^b} f(x)dx} = \underbrace{\lim_{n\to\infty} S_{\text{upper}}(f,n)}_{=\overline{\int_a^b} f(x)dx}.$$

The Riemann Integral

Alternative definition

The function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** if there exists a number $L = \int_a^b f(x)dx \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists some $n_0(f) > 0$, such that for $n > n_0(f)$, $|S(f, n) - L| < \varepsilon$ holds.

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This implies that if f is Riemann integrable, for any $\varepsilon > 0$ we can find $n_0(f)$ such that for $n > n_0(f)$:

$$\longrightarrow \underbrace{\int_a^b f(x)dx - S_{\text{lower}}(f, n)}_{>0} < \varepsilon$$

$$\longrightarrow \underbrace{S_{\text{upper}}(f, n) - \int_a^b f(x)dx}_{>0} < \varepsilon$$

Exercise 7.8

If f and g are Riemann integrable, let $S_{\text{lower}}(f,n)$, $S_{\text{upper}}(f,n)$, $S_{\text{lower}}(g,n)$ and $S_{\text{upper}}(g,n)$ be the lower and upper Riemann sums for f and g respectively when calculating $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ using n sub-intervals.

(i) What could you use for the lower and upper Riemann sums for $\int_0^1 (f(x) - g(x))\, dx$

(ii) Can you use a limiting procedure as $n \to \infty$ to prove that

$$\int_0^1 (f(x) - g(x))\, dx = \int_0^1 f(x)dx - \int_0^1 g(x)dx \,?$$

Additional exercise

(i) For $K > 0$, calculate

$$\int_{-K}^{K} x \exp\left(-\frac{1}{2}x^2\right) dx$$

(ii) Given that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2\right) dx = \sqrt{2\pi}$$

calculate

$$\int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2}(x-\mu)^2\right) dx$$

and

$$\int_{-\infty}^{\infty} (x-\mu)^2 \exp\left(-\frac{1}{2}(x-\mu)^2\right) dx.$$

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