Emilio Luis Sáenz Guillén
Doctoral Researcher in Actuarial Science

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Exercise 7.8 If f and g are Riemann integrable, let $S_{\text{lower}}(f,n)$, $S_{\text{upper}}(f,n)$, $S_{\text{lower}}(g,n)$ and $S_{\text{upper}}(g,n)$ be the lower and upper Riemann sums for f and g, respectively, when calculating $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ using the subintervals of n.

(i) What could you use for the lower and upper Riemann sums for $\int_0^1 (f(x)-g(x))\,dx$

(ii) Can you use a limiting procedure as $n\to\infty$ to prove that

$$\int_0^1 (f(x)-g(x))\,dx = \int_0^1 f(x)dx - \int_0^1 g(x)dx \ ?$$

Solution:

- f and g are Riemann-integrable over $[0,1]$ (note that this does not mean that $f-g$ is Riemann-integrable).
- $S_{\text{lower}}(f,n)$, $S_{\text{upper}}(f,n)$, $S_{\text{lower}}(g,n)$, $S_{\text{upper}}(g,n)$

Thus, what we are asking is similar to what the lecture notes provide, but in this case, for $a=1$.

(i) Over the generic interval $\left[\frac{i-1}{n},\frac{i}{n}\right]$ we know that:

$$\begin{cases} f_{i,\text{lower}} \le f(x) \le f_{i,\text{upper}} \\ g_{i,\text{lower}} \le g(x) \le g_{i,\text{upper}} \implies -g_{i,\text{upper}} \le -g(x) \le -g_{i,\text{lower}} \end{cases}$$

Therefore, over $\left[\frac{i-1}{n},\frac{i}{n}\right]$, it is also true that:

$$f_{i,\text{lower}} - g_{i,\text{upper}} \le f(x)-g(x) \le f_{i,\text{upper}} - g_{i,\text{lower}}.$$

Summing up $f_{i,\text{lower}} - g_{i,\text{upper}}$ over n (equidistant) sub-intervals we see that what we can use for the lower Riemann sum for $f-g$ is:

$$\frac{1}{n}\sum_{i=1}^{n}\left(f_{i,\text{lower}} - g_{i,\text{upper}}\right) \underset{\substack{\uparrow \\ \text{by linearity}}}{=} \frac{1}{n}\sum_{i=1}^{n} f_{i,\text{lower}} - \frac{1}{n}\sum_{i=1}^{n} g_{i,\text{upper}} = S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n)$$

Similarly, we see that what we can use for the upper Riemann sum for $f-g$ is:

$$\frac{1}{n}\sum_{i=1}^{n}\left(f_{i,\text{upper}} - g_{i,\text{lower}}\right) = S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n).$$

Let me briefly refer to the illustrative example that you have at the beginning of your lecture notes, that presented Riemann sums as the 'sum of rectangles': in here the width of each of our 'rectangles' is just the length of each of the n sub-intervals that we have over the interval $[0,1]$, which is equal to $\frac{1-0}{n}=\frac{1}{n}$; the height of each 'rectangle' is just $f_{i,\text{upper}} - g_{i,\text{lower}}$, for $i=1,...,n$.

(ii) By construction of the Riemann sums we always have that:

$$S_{\text{lower}}(f,n) \le \int_0^1 f(x)dx \le S_{\text{upper}}(f,n) \tag{1}$$

Indeed, note that if you always choose the smallest value of the function on each interval, the Riemann sum $S_{\text{lower}}(f,n)$ must be an underestimate of the Riemann integral $\int_0^1 f(x)dx$. If you choose the largest value of the function on each interval, you will get an overestimate, $S_{\text{upper}}(f,n)$, of $\int_0^1 f(x)dx$.

Moreover, because f is Riemann integrable we know that:

$$\begin{cases} \lim_{n\to\infty} S_{\text{lower}}(f,n) = \int_0^1 f(x)dx \\ \lim_{n\to\infty} S_{\text{upper}}(f,n) = \int_0^1 f(x)dx \end{cases}$$

In other words, and using the 'alternative definition' provided in the slides, for any $\varepsilon > 0$ there exists some $n_0(f)$, such that for $n > n_0(f)$:

$$\begin{cases} \int_0^1 f(x)dx - S_{\text{lower}}(f,n) < \varepsilon/2 \implies \int_0^1 f(x)dx - \varepsilon/2 < S_{\text{lower}}(f,n) \\ S_{\text{upper}}(f,n) - \int_0^1 f(x)dx < \varepsilon/2 \implies S_{\text{upper}}(f,n) < \int_0^1 f(x)dx + \varepsilon/2 \end{cases} \tag{2}$$

Hence, combining 1 and 2, for $n > n_0(f)$ we have that:

$$\int_0^1 f(x)dx - \varepsilon/2 < S_{\text{lower}}(f,n) \le \int_0^1 f(x)dx \le S_{\text{upper}}(f,n) < \int_0^1 f(x)dx + \varepsilon/2 \tag{3}$$

Similarly, because g is Riemann integrable , for any $\varepsilon > 0$ we can find $n_0(g)$ such that for $n > n_0(g)$:

$$\int_0^1 g(x)dx - \varepsilon/2 < S_{\text{lower}}(g,n) \le \int_0^1 g(x)dx \le S_{\text{upper}}(g,n) < \int_0^1 g(x)dx + \varepsilon/2$$

or,

$$-\int_0^1 g(x)dx - \varepsilon/2 < -S_{\text{upper}}(g,n) \le -\int_0^1 g(x)dx \le -S_{\text{lower}}(g,n) < -\int_0^1 g(x)dx + \varepsilon/2 \tag{4}$$

Therefore, combining 3 and 4, we can conclude that as long as $n > \max\{n_0(f), n_0(g)\}$[1], we have that:

$$\begin{aligned} \int_0^1 f(x)dx - \int_0^1 g(x)dx - \varepsilon < \underbrace{S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n)}_{\text{Riemann lower sum of } f-g} \le \\ \le \int_0^1 f(x)dx - \int_0^1 g(x)dx \\ \le \underbrace{S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n)}_{\text{Riemann upper sum of } f-g} < \int_0^1 f(x)dx - \int_0^1 g(x)dx + \varepsilon \end{aligned} \tag{5}$$

[1] Saying $n \ge \max\{n_0(f), n_0(g)\}$ is the same as saying 'for both $n > n_0(f)$ and $n > n_0(g)$'. And note 3 needs $n > n_0(f)$ to hold and 4 needs $n > n_0(g)$ to hold.

Note that we just got $\int_0^1 f(x)dx - \int_0^1 g(x)dx$ bounded by the Riemann lower and upper sums we derived in (i). Recall that the results obtained in (i) imply:

$$S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n) \leq \int_0^1 (f(x) - g(x)) \leq S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n) \tag{6}$$

Combine 5 and 6 and note that we obtain:

$$\int_0^1 f(x)dx - \int_0^1 g(x)dx - \varepsilon < \int_0^1 (f(x) - g(x)) < \int_0^1 f(x)dx - \int_0^1 g(x)dx + \varepsilon$$

that is,

$$\left| \int_0^1 (f(x) - g(x))\, dx - \left(\int_0^1 f(x)dx - \int_0^1 g(x)dx \right) \right| < \varepsilon$$

Hence, and since ε can be whatever positive number we want, letting $\varepsilon \to 0$, we get the desired result:

$$\int_0^1 f(x)dx - \int_0^1 g(x)dx = \int_0^1 (f(x) - g(x)).$$

Alternatively, we could also have reasoned as follows. Given that:

$$\begin{cases} S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n) \leq \int_0^1 (f(x) - g(x))\, dx \leq S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n) \\ S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n) \leq \int_0^1 f(x)dx - \int_0^1 g(x)dx \leq S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n) \end{cases}$$

we can take the limit on both sides and by Riemann integrability conclude that since:

$$\begin{cases} \lim_{n\to\infty} (S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n)) \underset{\substack{\uparrow \\ \text{by linearity of the lim}}}{=} \lim_{n\to\infty} S_{\text{lower}}(f,n) - \lim_{n\to\infty} S_{\text{upper}}(g,n) \underset{\substack{\uparrow \\ \text{by Riemann integrability}}}{=} S(f,n) - S(g,n) \\ \lim_{n\to\infty} (S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n)) \underset{\substack{\uparrow \\ \text{by linearity of the lim}}}{=} \lim_{n\to\infty} S_{\text{upper}}(f,n) - \lim_{n\to\infty} S_{\text{lower}}(g,n) \underset{\substack{\uparrow \\ \text{by Riemann integrability}}}{=} S(f,n) - S(g,n) \end{cases}$$

it must be the case that:

$$\implies \int_0^1 f(x)dx - \int_0^1 g(x)dx = \int_0^1 (f(x) - g(x)).$$

Additional exercise.

(i) For $K > 0$, calculate

$$\int_{-K}^{K} x \exp\left(-\frac{1}{2}x^2\right) dx$$

Note that,

$$\frac{d}{dx} - \exp\left(-\frac{1}{2}x^2\right) = -\frac{d}{dx} \underbrace{\exp\left(-\frac{1}{2}x^2\right)}_{=-x\exp\left(-\frac{1}{2}x^2\right)} = x \exp\left(-\frac{1}{2}x^2\right)$$

thus,

$$\int_{-K}^{K} x \exp\left(-\frac{1}{2}x^2\right) = \left[-\exp\left(-\frac{1}{2}x^2\right)\right]_{-K}^{K} = \exp\left(\underbrace{-\frac{1}{2}(-K)^2}_{<0}\right) - \exp\left(\underbrace{-\frac{1}{2}K^2}_{<0}\right) \xrightarrow{K\to\infty} 0$$

(ii) Given that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2\right) dx = \sqrt{2\pi}$$

calculate:

1.
$$\int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2}(x-\mu)^2\right) dx$$

Let us use the following substitution:

$$y = g(x) = x - \mu; \quad g(x = \infty) = \infty,$$
$$dy = dx; \quad g(x = -\infty) = -\infty$$

$$\int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2}(x-\mu)^2\right) dx = \int_{-\infty}^{\infty} (y+\mu) \exp\left(-\frac{1}{2}y^2\right) dy =$$
$$\underbrace{\int_{-\infty}^{\infty} y \exp\left(-\frac{1}{2}y^2\right) dy}_{=0 \text{ (see (i))}} + \mu \underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2\right) dy}_{=\sqrt{2\pi}} = \mu\sqrt{2\pi}.$$

2. Applying the same substitution as in 1:

$$\int_{-\infty}^{\infty} y^2 \exp\left(-\frac{1}{2}(x-\mu)^2\right) dx = \int_{-\infty}^{\infty} \underbrace{y^2}_{=y\times y} \exp\left(-\frac{1}{2}y^2\right) dy.$$

We will need to apply integration by parts. Consider,

$$u = y; du = dy$$
$$dv = y \exp\left(-\frac{1}{2}y^2\right) dy; v = -\exp\left(-\frac{1}{2}y^2\right)$$

then,

$$\int_{-\infty}^{\infty} y^2 \exp\left(-\frac{1}{2}(x-\mu)^2\right) dx = \int_{-\infty}^{\infty} \underbrace{y^2}_{=y\times y} \exp\left(-\frac{1}{2}y^2\right) dy =$$

$$= \left[-y \exp\left(-\frac{1}{2}y^2\right)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2\right) dy =$$

$$= \underbrace{\lim_{y\to-\infty} y \exp\left(-\frac{1}{2}y^2\right)}_{=0} - \underbrace{\lim_{y\to\infty} y \exp\left(-\frac{1}{2}y^2\right)}_{=0} + \sqrt{2\pi} = \sqrt{2\pi}.$$