AS1056 - Mathematics for Actuarial Science. Chapter 6, Tutorial 2.

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Some definitions...

Limit of a sequence

We call L the limit of the sequence (a_n), which is written $a_n \to L$, or $\lim_{n\to\infty} a_n = L$, if the following condition holds:
For any real number $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon)$ such that, for every natural number $n \geq n_0(\varepsilon)$, we have $|a_n - L| < \varepsilon$.

Contractive sequences

A sequence a_n is called contractive if there exists $k \in [0, 1)$ such that

$$|a_{n+2} - a_{n+1}| \leq k|a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}$$

Theorem: Every contractive sequence is convergent.

Exercise 6.7

A sequence is implicitly defined by the recursive equation $a_{n+1} = 16 + \frac{1}{2}a_n$ and has starting point $a_0 = 8$.

(i) Write down the values of a_n for $1 \leq n \leq 4$.

(ii) Identify the limit L of this sequence.

(iii) Define $b_n = a_n - L$. Write down an expression for b_n and, for $\varepsilon = 0.01$, find a value of n_0 such that $|b_n| < \varepsilon$ whenever $n \geq n_0$

- After showing that the above sequence is *contractive* and using the above theorem, it is quite straightforward to calculate the limit of the sequence.
- However, we can also calculate the limit of the sequence without knowing what a contractive sequence is, nor relying on any theorem. For such purposes, let me recall you the geometric series formulas:

$$s_n = \sum_{k=0}^{n-1} ar^k = \sum_{k=1}^{n} ar^{k-1} = \begin{cases} a\left(\frac{1-r^n}{1-r}\right), & \text{for } r \neq 1 \\ an, & \text{for } r = 1 \end{cases}$$

In particular for $n \to \infty$ we have that:

$$\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right) =$$

$$= \frac{a}{1-r}, \text{ for } |r| < 1$$

From some weeks ago...

Manipulating inequalities

Rule 1. Adding/subtracting the same quantity from both sides of an inequality leaves the inequality symbol unchanged.

Rule 2. Multiplying/dividing both sides by a positive number leaves the inequality symbol unchanged.

Rule 3. Multiplying/dividing both sides by a negative number reverses the inequality.

Rule 4. Applying any **monotonically increasing/decreasing** function to an inequality leaves the inequality symbol unchanged/ reverses the inequality. That is:

- $x \leq y \iff f(x) \leq f(y)$ if f is increasing.
- $x \leq y \iff f(x) \geq f(y)$ if f is decreasing.

In particular, raising both sides of an inequality to a power $n > 0$, when a and b are positive real numbers yields:

- $0 \leq a \leq b \iff 0 \leq a^n \leq b^n$
- $0 \leq a \leq b \iff a^{-n} \geq b^{-n} \geq 0.$

And raising both sides of an inequality to a power $n > 0$, when a and b are negative real numbers:

- $a \leq b \leq 0 \iff a^n \geq b^n \geq 0$
- $a \leq b \leq 0 \iff a^{-n} \leq b^{-n} \leq 0$

A couple of weeks ago, we said: 'Squaring both sides of an inequality if both sides are positive/negative leaves the inequality symbol unchanged/reverses the inequality.'
Indeed, note the square function is increasing for positive values of x and decreasing for negative values of x.

Exercise 6.1

(i) Write down the first 4 terms of the sequence $a_n = 2^{n-1}/(1+2^n)$.

(ii) What would you guess to be the limit, L?

(iii) In order to prove that L really is the limit, you need to come up with a way of choosing $n_0(\varepsilon)$ for every value of ε. How would you do this?

$\longrightarrow$ Note this exercise is somewhat similar to 6.7 though here the sequence is provided in explicit/closed form.

Exercise 6.10

(i) Explain why if $n \in \mathbb{N}$ and $x \leq n$, with $x > 0$, then $n^{-\frac{3}{2}} \leq \int_{n-1}^{n} x^{-\frac{3}{2}} dx$ holds for $n \geq 2$.

(ii) Use part (i) to find a value U such that $\sum_{n=2}^{\infty} n^{-\frac{3}{2}} \leq U$

(iii) Show that $\sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty$.

For (i) consider using the following rule for definite integrals:

Domination rule of definite integrals

$f(x) \geq g(x)$ on $[a, b]$ $\implies \int_b^a f(x)dx \geq \int_b^a g(x)dx$

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