Emilio Luis Sáenz Guillén
Doctoral Researcher in Actuarial Science

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Exercise 6.7 A sequence is implicitly defined by the recursive equation $a_{n+1} = 16 + \frac{1}{2}a_n$ and has starting point $a_0 = 8$.

(i) Write down the values of a_n for $1 \leq n \leq 4$.

$$\left.\begin{aligned} a_1 &= 16 + \frac{1}{2}a_0 = 20 \\ a_2 &= 16 + \frac{1}{2}a_1 = 26 \\ a_3 &= 16 + \frac{1}{2}a_2 = 29 \\ a_4 &= 16 + \frac{1}{2}a_3 = 30.5 \\ &\vdots \end{aligned}\right\} \quad a_2 - a_1 = 6, \quad a_3 - a_2 = 3, \quad a_4 - a_3 = 1.5$$

(ii) Identify the limit L of this sequence.

First, let us show that $(a_n)_{n\in\mathbb{N}}$ is contractive and then we'll apply the theorem in the tutorial slides. That is, show there exists $k \in [0, 1)$ s.t. $|a_{n+2} - a_{n+1}| \leq k|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$.

Proof.

$$|a_{n+2} - a_{n+1}| = \left|\cancel{16} + \frac{1}{2}a_{n+1} - \cancel{16} - \frac{1}{2}a_n\right| = \left|\frac{1}{2}(a_{n+1} - a_n)\right| =$$
$$= \frac{1}{2}\cancel{|a_{n+1} - a_n|} \leq k \times \cancel{|a_{n+1} - a_n|}; \quad \frac{1}{2} \leq k < 1$$

and the proof is done since we've shown that such a k exists, in particular $k \in \left[\frac{1}{2}, 1\right)$. □

Now, note that every contractive sequence is convergent (see theorem in tutorial notes) and thus we can be sure that L exists, i.e., that the sequence has a limit. Thus, taking the limit on both sides of our recursive equation, this simplifies to:

$$\underbrace{\lim_{n\to\infty} a_{n+1}}_{=L} = 16 + \frac{1}{2}\underbrace{\lim_{n\to\infty} a_n}_{=L}$$
$$L = 16 + \frac{1}{2}L; \quad L = 32$$

Alternatively —and without using any theorem—, we could have considered the following approach. Note,

$$\begin{aligned}
a_0 &= 8 \\
a_1 &= 16 + \frac{1}{2}a_0 \\
a_2 &= 16 + \frac{1}{2}a_1 = 16 + \frac{1}{2}\left(16 + \frac{1}{2}a_0\right) = 16 + 8 + \left(\frac{1}{2}\right)^2 a_0 \\
a_3 &= 16 + \frac{1}{2}a_2 = 16 + \frac{1}{2}\left[16 + 8 + \left(\frac{1}{2}\right)^2 a_0\right] = 16 + 8 + 4 + \left(\frac{1}{2}\right)^3 a_0 = \\
&= \sum_{k=0}^{2} \frac{16}{2^k} + \left(\frac{1}{2}\right)^3 a_0 = 16\sum_{k=0}^{2}\left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^3 a_0 \\
&\vdots \\
a_n &= 16\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^n a_0
\end{aligned}$$

$\longrightarrow$ We've converted this "implicit definition of the recurrence" that we were provided into a closed, explicit formula. This is usually referred to as *solving a recurrence relation*.

And now, given the above explicit expression (and using geometric series formula), we can easily calculate the limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 16\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^n a_0 = 16\underbrace{\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k}_{=\frac{1}{1-\frac{1}{2}}=2} + \underbrace{\underbrace{\lim_{n\to\infty}\left(\frac{1}{2}\right)^n}_{\to 0} a_0}_{\to 0} = 32$$

(iii) Define $b_n = a_n - L$. Write down an expression for b_n and, for $\varepsilon = 0.01$, find a value of n_0 such that $|b_n| < \varepsilon$ whenever $n \geq n_0$.

In other words, we're asked to apply the definition of limit of a sequence to prove that, indeed, the limit of (a_n) is $L = 32$.

Let us start writing,

$$b_n = a_n - \overbrace{L}^{=32}, \text{ i.e., } a_n = 32 + b_n$$

thus, the recursive equation for (b_n) is:

$$b_{n+1} = a_{n+1} - 32 = 16 + \frac{1}{2}a_n - 32 = \frac{1}{2}a_n - 16 = \frac{1}{2}(32 + b_n) - 16 = \cancel{16} + \frac{1}{2}b_n - \cancel{16} = \frac{1}{2}b_n.$$

And recursively applying this recursive equation:

$$b_{n+1} = \frac{1}{2}\underbrace{b_n}_{=\frac{1}{2}\underbrace{b_{n-1}}_{=\frac{1}{2}\underbrace{b_{n-2}}_{\vdots}}} \quad \left.\right\} n \text{ times}$$

we get that,

$$b_n = \left(\frac{1}{2}\right)^n \underbrace{b_0}_{\substack{a_0-L= \\ =8-32=-24}} = -24 \times \left(\frac{1}{2}\right)^n$$

Now that we have (b_n) in explicit form, from the definition of the limit of a sequence, to prove that $L = 32$, we need to show that for any $\varepsilon > 0$ we can find some $n_0(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq n_0(\varepsilon)$, $|a_n - L| = |b_n| < \varepsilon$ holds.

So, we require,

$$|b_n| < \varepsilon; \quad \underbrace{\left|-24\left(\frac{1}{2}\right)^n\right|}_{=\left|24\left(\frac{1}{2}\right)^n\right|} < \varepsilon; \quad 24\left(\frac{1}{2}\right)^n < \varepsilon; \quad \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{24};$$

↑ by symmetry property of abs. values

then,

$$n \underbrace{\ln\left(\frac{1}{2}\right)}_{=\ln(1)-\ln(2)=0-\ln(2)} < \ln\left(\frac{\varepsilon}{24}\right); \quad -n\ln(2) < \ln\left(\frac{\varepsilon}{24}\right); \quad n\ln(2) > -\ln\left(\frac{\varepsilon}{24}\right);$$

$$n > -\frac{1}{\ln(2)}\ln\left(\frac{\varepsilon}{24}\right) \underset{\text{for } \varepsilon = 0.01}{\uparrow}{=} 11.22881869$$

$\longrightarrow$ Thus, $n_0(\varepsilon = 0.01) = 12$. In other words, $|a_n - L| = |b_n| < \varepsilon = 0.01$ holds as long as $n \geq n_0(\varepsilon = 0.01) = 12$. And you can consider other values of ε, replace in $-\frac{1}{\ln(2)}\ln\left(\frac{\varepsilon}{24}\right)$, and then get $n_0(\varepsilon)$ to check that "for every natural number $n \geq n_0(\varepsilon)$, we have $|a_n - L| < \varepsilon$". And this is basically the proof that $L = 32$.

Exercise 6.10

(i) Explain why if $n \in \mathbb{N}$ and $x \leq n$, with $x > 0$, then $n^{-\frac{3}{2}} \leq \int_{n-1}^{n} x^{-\frac{3}{2}} dx$ holds for $n \geq 2$.

So, note that:

$$x \leq n \implies x^{3/2} \leq n^{3/2}; \quad n^{-3/2} \leq x^{-3/2}$$

↑ applying a monotonically increasing function on both sides of the inequality leaves the inequality unchanged

By the domination rule of definite integrals:

$$n^{-3/2} \leq x^{-3/2} \implies \underbrace{\int_n^{n-1} n^{-3/2} dx}_{n^{-3/2} \underbrace{\int_n^{n-1} 1dx}_{=[x]_{n-1}^{n}=n-n+1=1} = n^{-3/2}} \leq \int_n^{n-1} x^{-3/2} dx$$

Since it is not clear whether we are considering $\mathbb{N}$ with or without 0, let's think why do we need $n \geq 2$?
$\longrightarrow x > 0 \implies n-1 > 0,$ i.e., $n > 1$, and since $n \in \mathbb{N}$, then $n \geq 2$.

(ii) Use part (i) to find a value U such that $\sum_{n=2}^{\infty} n^{-\frac{3}{2}} \leq U$.

From (i) we have,

$$n^{-3/2} \leq \int_n^{n-1} x^{-3/2} dx$$

then,

$$\sum_{n=2}^{\infty} n^{-3/2} \leq \sum_{n=2}^{\infty} \int_n^{n-1} x^{-3/2} dx = \int_1^{\infty} x^{-3/2} dx = \left[\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right]_1^{\infty} = \left[-2x^{-\frac{1}{2}} \right]_1^{\infty} =$$
$$= \underbrace{2 \times 1^{-\frac{1}{2}}}_{=2} - \underbrace{\lim_{x \to \infty} 2x^{-\frac{1}{2}}}_{=0} = 2$$

(iii) Show that $\sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty$.

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}} = \underbrace{\sum_{n=2}^{\infty} n^{-\frac{3}{2}}}_{\leq 2} + 1 \leq 3.$$