AS1056 - Mathematics for Actuarial Science. Chapter 5, Tutorial 2.

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Exercise 5.8

$£1,000$ is invested at time 0 and earns interest *continuously* at a fixed rate of 6%, so that the value of the investment at time t is $x(t) = £1,000 \times 1.06^t$. The investment is sold at time T years. Which of the following statements are correct?

(i) Investing $£2,000$ for $\frac{1}{2}T$ years would have resulted in a larger profit.

(ii) Investing $£500$ for T years at a rate of 12% would have resulted in a larger profit.

(iii) It is possible to find an interest rate r with the property that a sum of $£750$ invested at rate $r\%$ for $1.5T$ years gives the same return as an investment of the sum of $£1,000$ invested at 6% for T years.

Compound interest

The total accumulated value, including the principal sum P plus compounded interest I, is given by the formula:

$$A = P\left(1 + \frac{r}{n}\right)^{tn}$$

where:

- A is the final amount
- P is the original principal sum
- r is the nominal annual interest rate
- n is the compounding frequency
- t is the overall length of time the interest is applied (expressed using the same time units as r, usually years).

And if the compounding frequency tends to ∞:

$$A = P \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{tn} = P \cdot e^{rt}$$

where:

- A is the amount of money accumulated after t years, including interest.
- P is the initial principal (in this case, £1,000).
- r is the continuous interest rate (here, 6% or 0.06 as a decimal).
- t is the time in years.
- And a common definition of the number e is: $\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}$.

Exercise 5.10

(i) Calculate the derivative of $f(x) = x^{-1}\ln(x) = \frac{\ln(x)}{x}$ over the domain $x > 0$.

(ii) Sketch the graph of f.

(iii) For which values of x is there more than one value of y which satisfies the equation $x\ln(y) = y\ln(x)$?

(iv) For which values of x does the equation $x\ln(y) = 2y\ln(x)$ have:

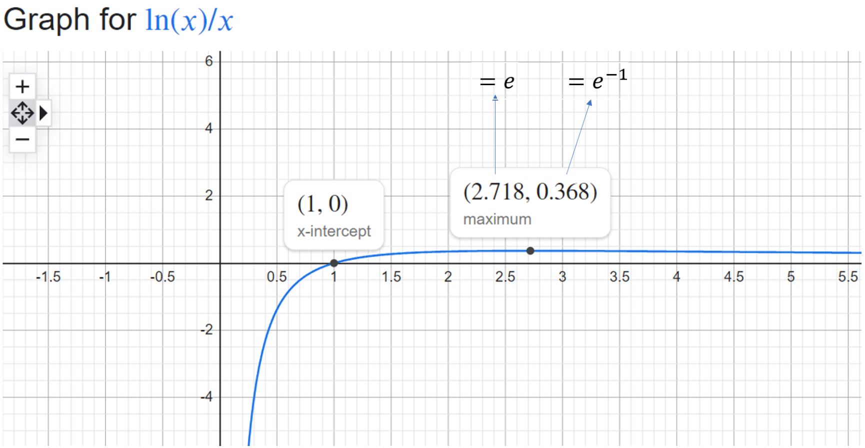
  (a) no solutions
  (b) one solution
  (c) two solutions?

(i) Calculate the derivative of $f(x) = x^{-1}\ln(x) = \frac{\ln(x)}{x}$ over the domain $x > 0$.

Answer:

$$f'(x) = -x^{-2}\ln(x) + x^{-1}x^{-1} = -x^{-2}\ln(x) + x^{-2} =$$
$$= \frac{1}{x^2}\left[1 - \ln(x)\right]$$

(ii) Sketch the graph of f.



Let us check analytically that:

1. "As $x \to 0^+$, $f(x) \to -\infty$."
2. "f first reaches 0 at $x = 1$."
3. "f has a maximum at $x = e$."
4. "$f(x)$ is increasing for $x \in (0, e)$ and decreasing for $x \in (e, +\infty)$."
5. "$\lim_{x \to +\infty} f(x) = 0$."

Summary

The function $f(x)$ is defined for all x in the interval $(0, +\infty)$. It increases from $-\infty$ to e^{-1} as x moves from 0 to e. $f(x)$ has a root at $x = 1$ and at $x = e$, $f(x)$ achieves its maximum value of e^{-1}. Then it decreases to 0 as x goes from e to $+\infty$.

(iii) For which values of x is there more than one value of y which satisfies the equation $x \ln(y) = y \ln(x)$?

Note that we can rewrite $x \ln(y) = y \ln(x)$ as $\frac{\ln(y)}{y} = \frac{\ln(x)}{x}$, i.e., as $f(y) = f(x)$, thus:

- It is clear that $y = x$ is always a solution
- Moreover, based on the properties of $f(x)$ that we have just discussed we'll be able to describe the behaviour of this new equation too.

(iv) For which values of x does the equation $x\ln(y) = 2y\ln(x)$ have:

(a) no solutions

(b) one solution

(c) two solutions?

Let us rewrite $x\ln(y) = y\ln(x)$ as $\frac{\ln(y)}{y} = 2 \times \frac{\ln(x)}{x}$, i.e., $f(y) = 2 \times f(x)$ or $f(x) = \frac{1}{2}f(y)$.

Reconsider the intervals for x we've been analysing thus far:

- $x \in (0, 1]$
- $x \in (1, e)$ and $x \in (e, +\infty)$ and $x = e$

Exercise 5.5

Use the formula $\cos\left(\frac{\pi}{5}\right) = -\cos\left(2 \times \frac{2\pi}{5}\right)$ to obtain a cubic equation satisfied by the value of $\cos\left(\frac{\pi}{5}\right)$.

For this exercise keep in mind the following trigonometric identities:

Trigonometric Identities

- $\cos((\pi - x)) = -\cos(x) \implies \cos\left(\frac{\pi}{5}\right) = -\cos\left(2 \times \frac{2\pi}{5}\right)$ for $x = \frac{4\pi}{5}$
- $\cos(2x) = 2\cos^2(x) - 1$

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