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AS1056 - Chapter 5, Tutorial 2. 14-11-2024. Notes.

Hello everyone, I'd like to go over the exercise we discussed today, and provide a detailed, step-by-step solution, especially for the more challenging sub-items (iii) and (iv). By meticulously working through the initial sub-items, I believe we can successfully tackle the entire exercise. So let's get to it.

Exercise 5.10

(i) Calculate the derivative of $f(x) = x^{-1}\ln(x) = \frac{\ln(x)}{x}$ over the domain $x > 0$.

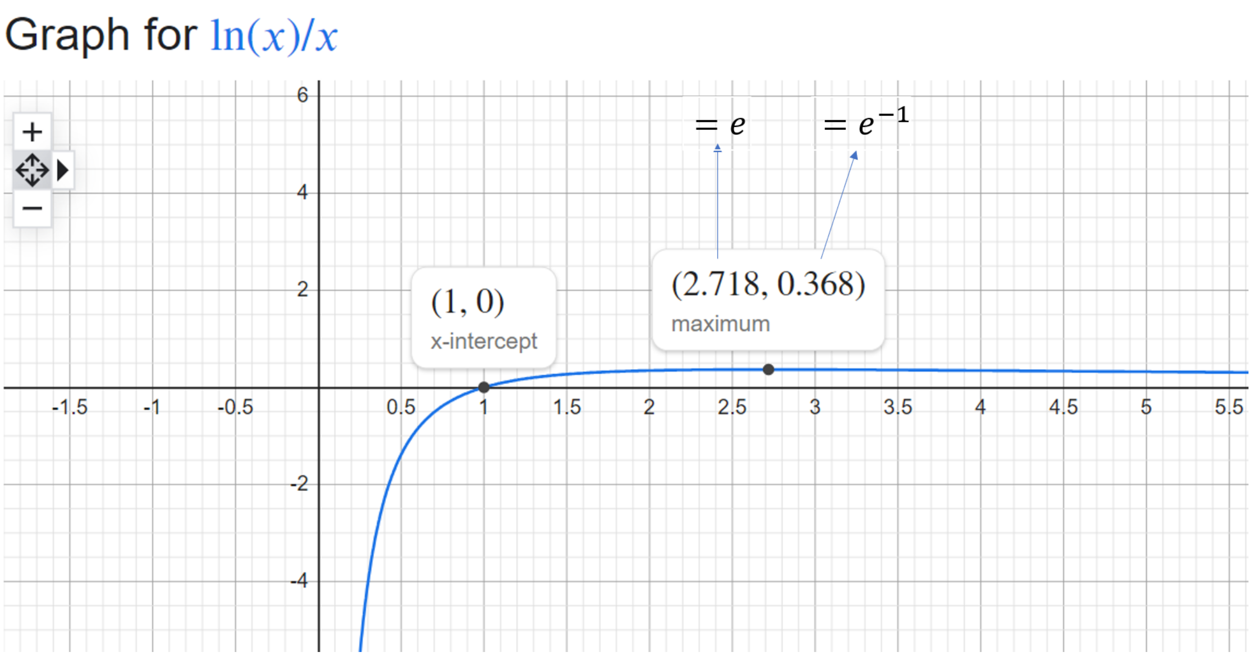
Solution:

$$f'(x) \underset{\uparrow}{=} \frac{(1/x)x - \ln x}{x^2} = \frac{1}{x^2}\left[1 - \ln(x)\right]$$

quotient rule for differentiation

(ii) Sketch the graph of f.

Solution: Use your favourite graphing calculator to plot $f(x) = \frac{\ln(x)}{x}$:



As you can observe the function $f(x)$ is defined for all x in the interval $(0, +\infty)$. It increases from $-\infty$ to e^{-1} as x moves from 0 to e. $f(x)$ has a root at $x = 1$ and achieves a maximum value of e^{-1} at $x = e$. Then it decreases very smoothly to 0 as x goes from e to $+\infty$.

Can we characterise the behaviour of $f(x)$ analytically without relying on its graph? Let's attempt to derive a description comparable to the visual interpretation we have just provided by proving the following properties analytically. Doing so will deepen our understanding of $f(x)$'s characteristics, and help us in addressing sub-items (iii) and (iv) of the exercise:

1. **"As $x \to 0^+$, $f(x) \to -\infty$."** Recall that $\ln(x)$ is defined only for $x > 0$, hence we should look at the limit as $x \to 0^+$, i.e., as x approaches 0 from the right (indeed, because there are no values to the left of 0 in the domain of $\ln x$, the limit $\ln x \to 0^-$ does not exist, and consequently, the limit $x \to 0$ does not exist either):

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty$$

Why $\frac{-\infty}{0^+} = -\infty$? Think about $\frac{-\infty}{0^+}$: on the one hand, a very big number over a very small number will be equal to a very big number; on the other hand, a negative number over a positive number will equal a negative number. For a more rigorous solution we would probably need to use the definition of the limit (ε's and δ's...).

If you don't have in mind that $\lim_{x\to 0^+} \ln(x) = -\infty$, we can always show it. Note that,

$\ln(x) = \ln\left(\frac{1}{1/x}\right) = \underbrace{\ln(1)}_{=0} - \ln\left(\frac{1}{x}\right) = -\ln\left(\frac{1}{x}\right)$. It is clear that $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\implies \lim_{x\to 0^+} \ln\left(\frac{1}{x}\right) = +\infty$ then, $\lim_{x\to 0^+} \ln(x) = \lim_{x\to 0^+} -\ln\left(\frac{1}{x}\right) =$

$\underset{\uparrow}{=} -\lim_{x\to 0^+} \ln\left(\frac{1}{x}\right) = -\infty$ (I hope it is clear that $\lim_{x\to +\infty} \ln(x) = +\infty$)

by linearity property of the limits

2. **"f first reaches 0 at $x = 1$."** We just need to find the root(s) of x:

$$f(x) = \frac{\ln(x)}{x} = 0;\ \ln(x) = 0;\ e^{\ln(x)} = e^0 = 1;\ x = 1$$

3. **"f has a maximum at $x = e$."**

$\longrightarrow$ First we find the critical points by setting $f'(x) = 0$ and solving for x:

$$\frac{1}{x^2}\left[1 - \ln(x)\right] = 0;\ 1 - \ln(x) = 0;\ \ln(x) = 1;\ e^{\ln(x)} = e^1;\ x = e^1 = e$$

At $x = e$, $f(x)$ takes the value:

$$f(x = e) = \frac{\ln(e)}{e} = \frac{1}{e} = e^{-1}$$

Thus, we have an critical point at $(x = e, y = e^{-1})$, and checking that $f''(x = e) < 0$ we conclude that $f(x)$ has a local maximum at (e, e^{-1}).

$\longrightarrow$ since f is a continuous real function with a unique local maximum and no other local extremum, we conclude that, indeed, $(x = e, y = e^{-1})$ is the absolute maximum of $f(x)$.[1]

[1] Unique Local Extremum is Absolute Extremum for Continuous Functions.

4. **"$f(x)$ is increasing for $x \in (0, e)$ and decreasing for $x \in (e, +\infty)$."**
   Note that this is a direct consequence from 3 (unique maximum, thus to the left of this maximum the functions needs to be increasing and to the right of this maximum it needs to be decreasing). However, we can always check the sign of the first derivatives for the above intervals:

   - $f(x)$ is increasing for $x \in (0, e)$, i.e., $f'(x) > 0$ for $x \in (0, e)$. Note that for $x \in (0, e)$:

     $$f'(x) = \underbrace{\frac{1}{x^2}}_{>0} \underbrace{[1 - \overbrace{\ln(x)}^{<1}]}_{>0} \longrightarrow \frac{1}{x^2} > 0 \text{ always, while } \ln(x) < 1 \text{ for } x \in (0, e).$$

     Thus, $f'(x) > 0$ for $x \in (0, e)$.

   - $f(x)$ is decreasing for $x \in (e, +\infty)$, i.e., $f'(x) < 0$ for $x \in (e, +\infty)$. Note that for $x \in (e, +\infty)$:

     $$f'(x) = \underbrace{\frac{1}{x^2}}_{>0} \underbrace{[1 - \overbrace{\ln(x)}^{>1}]}_{<0} \longrightarrow \frac{1}{x^2} > 0 \text{ always, while } \ln(x) > 1 \text{ for } x \in (e, +\infty).$$

     Thus, $f'(x) < 0$ for $x \in (e, +\infty)$.

   Note: If you don't see why $\ln(x) < 1$ for $x \in (0, e)$ and $\ln(x) > 1$ for $x \in (e, +\infty)$ just recall that $\ln(e) = 1$ and that $\ln(x)$ is an increasing function[2].

5. **"$\lim_{x\to+\infty} f(x) = 0$."**

   $$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln(x)}{x} \underset{\substack{\uparrow \\ \text{l'Hôpital}}}{=} \lim_{x\to+\infty} \frac{1/x}{1} = 0$$

(iii) For which values of x is there more than one value of y which satisfies the equation $x\ln(y) = y\ln(x)$?

Solution: Note that we can rewrite $x\ln(y) = y\ln(x)$ as $\frac{\ln(y)}{y} = \frac{\ln(x)}{x}$, i.e., as $f(y) = f(x)$, thus:

- The equation $f(y) = f(x)$ is symmetric in x and y: any point (x, y) that lies on the curve will also have its reflection point (y, x) lie on the curve.
- When $x = y$, the equation $f(y) = f(x)$ is trivially satisfied.
- Moreover, based on the properties of $f(x)$ that we have just discussed we'll be able to describe the behaviour of this new equation too. In particular, to determine for which values of x there is more than one value of y satisfying the equation $x\ln(y) = y\ln(x)$, we just need to identify the intervals of x where there is more than one value of x satisfying $y = f(x)$.

Hence, given that we have already thoroughly worked out the properties of $f(x)$ on the previous sub-items, I will suggest you to answer this sub-item analytically (using what we

[2] $\frac{\partial \ln(x)}{\partial x} = \frac{1}{x} > 0$ since $\ln(x)$ is only defined for $x > 0$.

know about $f(x)$), and afterwards check the conclusions obtained sketching a graph of the equation.

Recall that $f(x)$ is a function that starts from $-\infty$ when x is close to 0, crosses the x-axis at $x = 1$, achieves its maximum value at $x = e$ and then smoothly decays to 0 as x grows larger and larger. Let us then consider the behaviour of $f(x)$ —and then infer the behaviour of $f(y) = f(x)$—, on the following intervals of x: $(0, 1]$; $(1, e)$ and $(e, +\infty)$; $\{e\}$.

1. For $x \in (0, 1]$ there's no other solution rather than $y = x$, since we know that:
   - $f(x)$ has a root at $x = 1$
   - $f(x) \leq 0$ for $x \in (0, 1]$ and $f(x) > 0$ for $x \in (1, +\infty)$
   - $f'(x) > 0$ for $x \in (0, 1]$ (i.e., monotonicity on (0,1])

   For x within $(0, 1]$, the function $f(x) = \frac{\ln(x)}{x}$ is negative and increasing, where $f(x)$ crosses from negative to positive at $x = 1$. As a result, there is no $x_1 > 1$ such that $f(x_1) = f(x_0)$ for x_0 in $(0, 1]$ since $f(x)$ is negative for $x < 1$ and positive for $x > 1$. In addition, the monotonic nature of $f(x)$ in $(0, 1]$ tells us that $f(x)$ is a one-to-one function in this interval. In other words, since $f(x)$ is increasing for $x \in (0, 1]$ we know that there's no $x_0' \neq x_0$, $x_0' \in (0, 1]$, such that $f(x_0') = f(x_0)$. Therefore, for x in $(0, 1]$, $f(y) = f(x)$ is also a one to one mapping, i.e., the only pair that satisfies the equation $f(y) = f(x)$ for each x in $(0, 1]$ is when y is exactly equal to x. This means that the equation $x \ln(y) = y \ln(x)$ has a unique solution $y = x$ in this interval.
2. $x \in (1, e)$ and $x \in (e, +\infty)$.
   - $x \in (1, e) \implies$
     $f(x) \in \left(f(1) = 0, f(e) = e^{-1}\right);\ f'\left((1, e)\right) > 0$ (increasing)
   - $x \in (e, +\infty) \implies$
     $f(x) \in \left(f(e) = e^{-1}, \lim_{x \to +\infty} f(x) = 0\right);\ f'\left((e, +\infty)\right) < 0$ (decreasing)

   Given the above and that we know there's a maximum at $x = e$ it is clear that for every $x_0 \in (1, e)$ there exists one $x_1 \in (e, +\infty)$ such that $f(x_0) = f(x_1)$. Thus:
   - ➢ For values $x_0 \in (1, e)$ there are two values of y that satisfy the equation $x \ln(y) = y \ln(x)$:
     - $y_0 \in (1, e)$, in fact $y_0 = x_0$, and,
     - $y_1 \in (e, +\infty)$
   - ➢ And vice versa, for values $x_1 \in (e, +\infty)$ there are two values of y that satisfy the equation $x \ln(y) = y \ln(x)$:
     - $y_0 \in (1, e)$, and,
     - $y_1 \in (e, +\infty)$, in fact $y_1 = x_1$
3. For $x \in \{e\}$, i.e., $x = e$, there's no other solution rather than $y = x$.

   $\longrightarrow f(x = e) = \dfrac{\ln(e)}{e} = \dfrac{1}{e} = e^{-1}$ and the only y value such that,
   $f(y) = f(x = e) = e^{-1}$ is clearly $y = e$.

(iv) For which values of x does the equation $x \ln(y) = 2y \ln(x)$ have: (a) no solutions, (b) one solution, (c) two solutions?

Solution: Let us rewrite $x \ln(y) = y \ln(x)$ as $\frac{\ln(y)}{y} = 2 \times \frac{\ln(x)}{x}$, i.e., $f(y) = 2 \times f(x)$ or $f(x) = \frac{1}{2} f(y)$.

Reconsider the intervals for x we've been analysing thus far:

1. $x \in (0, 1]$
2. $x \in (1, e)$ and $x \in (e, +\infty)$ and $x = e$

1. As we have checked in the previous sub-items for $x \in (0, 1]$ there is one and only one solution for $f(x)$.
   $\implies$ given that $f(y) = 2f(x)$, then, there is also one and only one solution for $f(y)$. In other words, for $x \in (0, 1]$ there is one and only one y —and you can check that $y \in (0, 1]$ for $x \in (0, 1]$[3]—, which satisfies $x \ln(y) = 2y \ln(x)$.
2. $x \in (1, e)$ and $x \in (e, +\infty)$ and $x = e$
   Note that taken by separate both $f(x)$ and $f(y)$ range from $-\infty$ up to e^{-1}. However, going back to $f(y) = 2f(x)$ we notice that:
   - If $f(x) = -\infty \implies f(y) = 2 \times (-\infty) = -\infty$, which is fine.
   - Nevertheless, if $f(x) = e^{-1} \implies f(y) = 2e^{-1}$, but this cannot happen since $f(y) \in (-\infty, e^{-1}]$

   So, implicitly, $f(y) = 2f(x)$, i.e., $f(x) = \frac{1}{2}f(y)$ is telling us that for $x \ln(y) = 2y \ln(x)$ to hold we need that $f(x)$ ranges from $-\infty$ up to $\frac{1}{2}e^{-1}$. And of course, given the characteristics of $f(x)$ that we have already studied, we know that there will be two values of x such that $f(x) = \frac{1}{2}e^{-1}$. In particular, there is one value $x_0 \in (1, e)$ such that $f(x_0) = \frac{1}{2}e^{-1}$, and another value $x_1 \in (e, +\infty)$ such that $f(x_1) = \frac{1}{2}e^{-1}$.[4]

   Hence, $f(y) = 2 \times f(x) \implies f(x) \in \left(-\infty, \frac{1}{2}e^{-1}\right]$. So, let us redefine the intervals of x on which to analyse the behaviour of $f(y) = 2 \times f(x)$:

   (1) $1 < x < x_0 = 1.261070487$
   (2) $x_0 = 1.261070487 < x < x_1 = 14.56100391$
   (3) $x > x_1 = 14.56100391$
   (4) $x = x_0 = 1.261070487$; $x = x_1 = 14.56100391$

   And let us draw some conclusions as follows:

   (2) If $x \in (x_0, x_1)$ there is no solution to the equation since $f(y)$ cannot exceed e^{-1}.
   (4) $x = x_0 \implies f(x_0) = \frac{1}{2}e^{-1}$ and $x = x_1 \implies f(x_1) = \frac{1}{2}e^{-1}$. In this case there is only one value of y that makes $f(x) = \frac{1}{2}f(y)$ hold. This value is $y = e$, which makes $f(y) = e^{-1}$.
   (1) and (3): for $1 < x < x_0$ and $x > x_1$ there are two values of y that satisfy the equation $f(y) = 2f(x)$. This because of the characteristics of $f(x)$ and that in these range of values of x no violation of the range of $f(y)$ occurs.

Remember to plot the equations of sub-items (iii) and (iv) and check the results!

[3] $\lim_{x \to 0^+} f(x) = -\infty \implies f(y) = 2 \times (-\infty) \implies y \to 0^+$ and $f(x = 1) = 0 \implies f(y) = 2 \times 0 \implies y = 1$.

[4] The equation $\frac{1}{2}e^{-1} = \frac{\ln(x)}{x}$ has not closed form solution, but you can approximate the values of x_0 and x_1 using some approximation method such as Newton-Raphson (we'll see this on the 2nd term). Specifically, $x_0 = 1.261070487$ and $x_1 = 14.56100391$.

Exercise 5.8 £$1,000$ is invested at time 0 and earns interest *continuously* at a fixed rate of 6%, so that the value of the investment at time t is $x(t) = £1,000 \times 1.06^t$. The investment is sold at time T years. Which of the following statements are correct?

$$x(t) = 1,000 \times \underbrace{1.06^t}_{=(1+r)^t} = 1,000 \times \underbrace{e^{t\ln(1.06)}}_{e^{tr}}$$

$\longrightarrow$ Annual compounding at a fixed nominal annual rate $r = 0.06$ is equivalent to continuous compounding at a fixed nominal annual rate $r = \ln(1.06)$.

(i) Investing £$2,000$ for $\frac{1}{2}T$ years would have resulted in a larger profit.

$$2,000 \times 1.06^{T/2} > 1,000 \times 1.06^T?$$

This can be rewritten as,

$$e^{\ln\left(2,000\times 1.06^{T/2}\right)} > e^{\ln\left(1,000\times 1.06^{T}\right)}$$

therefore, we want to know whether

$$\ln(2,000) + \frac{T}{2}\ln(1.06) > \ln(1,000) + T\ln(1.06)?$$

subtracting $\ln(1,000)$ on both sides:

$$\ln\left(\frac{2,000}{1,000}\right) + \cancel{\frac{T}{2}\ln(1.06)} > T\ln(1.06) = \frac{T}{2}\ln(1.06) + \cancel{\frac{T}{2}\ln(1.06)}$$

in conclusion the inequality will hold as long as:

$$\ln(2) > \frac{T}{2}\ln(1.06); \quad 2\ln(2) > T\ln(1.06); \quad T < \frac{2\ln(2)}{\ln(1.06)} = 23.79132209.$$

(ii) Investing £500 for T years at a rate of 12% would have resulted in a larger profit. Check solutions on Moodle!

(iii) It is possible to find an interest rate r with the property that a sum of £750 invested at rate r% for $1.5T$ years gives the same return as an investment of the sum of £$1,000$ invested at 6% for T years.

$$750 \times (1+r)^{1.5T} = 1,000 \times 1.06^T; \quad e^{\ln\left(750\times(1+r)^{1.5T}\right)} = e^{\ln\left(1,000\times 1.06^T\right)}$$

$$\ln(750) + 1.5T \times \ln(1+r) = \ln(1,000) + T\ln(1.06); \quad 1.5T\ln(1+r) = \ln\left(\frac{4}{3}\right) + T\ln(1.06)$$

$$\ln(1+r) = \frac{1}{1.5T}\ln\left(\frac{4}{3}\right) + \frac{\ln(1.06)}{1.5}; \quad 1+r = \left(\frac{4}{3}\right)^{\frac{1}{1.5T}} \times 1.06^{1/1.5}$$

$$r = 1.06^{2/3} \times \left(\frac{4}{3}\right)^{\frac{2}{3T}} - 1.$$

From this expression we see that mathematically speaking we need $T \neq 0$. And since, it doesn't make sense to have negative time, we conclude that $T > 0$.