AS1056 - Mathematics for Actuarial Science. Chapter 3, Tutorial 2.

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1. Review

2. Exercises

Big-O notation

- **Purpose:** Describe an upper bound on the time complexity of an algorithm in terms of the *worst-case scenario*.
- **Usage:** Commonly used in computer science to analyse the efficiency of algorithms.
- **Example:** If an algorithm has a time complexity of $O(n^2)$, it means that the number of operations performed by the algorithm grows at most quadratically with respect to the size of the input.

```r
sort_List <- function(List) {
  n <- length(List)
  for (i in 1:(n - 1)) {
    min_index <- i

    for (j in (i + 1):n) {
      if (List[j] < List[min_index]) min_index <- j
    }

    # Swap the ith element with the smallest found in the unsorted portion
    if (min_index != i) {
      temp <- List[i]
      List[i] <- List[min_index]
      List[min_index] <- temp
    }
  }
  return(List)
}
```

runs $n - i$ times

runs $n - 1$ times

$\sum_{i=1}^{n-1} n - i = n - 1 + n - 2 + \ldots + 1 = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} = O(n^2)$ since the total number of operations (iterations) grows $\leq$ faster than n^2.

Definition (I)

Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. We say that,

$$f(x) = O(g(x)) \text{ as } x \to \infty$$

if there is **at least one** choice of a constant $M > 0$, for which you can find a constant k such that:

$$|f(x)| \leq M|g(x)| \quad \text{i.e.} \quad \left|\frac{f(x)}{g(x)}\right| \leq M$$

whenever $x > k$. Beyond some point k, function $f(x)$ is at most a constant M times $g(x)$.

$\longrightarrow f(x) = O(g(x))$ (big-oh) if *eventually* (namely when $x > k$), f grows slower than **some** multiple of g.

We can also use this notation to describe the behaviour of a function nearby **some** real number a (often $a = 0$).

Definition (II)

We say that,

$$f(x) = O(g(x)) \text{ as } x \to a$$

if there is **at least one** constant M such that,

$$\left|\frac{f(x)}{g(x)}\right| \leq M$$

for sufficiently small x.

The intuition behind big-oh notation is that f is $O(g)$ if $g(x)$ grows as fast or faster than $f(x)$ as $x \to a$.

Little-o notation

- **Purpose:** Describes an upper bound, but in a stronger sense than big-O, indicating that a function grows strictly slower than the comparison function.
- **Usage:** Less common than Big O, but used when we need to express that one function grows strictly slower than another.
- **Example:** If $f(n) = n$ and $g(n) = n^2$ then $f(n) = o(g(n))$ as $n \to \infty$ because $f(n)$ grows strictly slower than $g(n)$.

Definition (I)

Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. We say that,

$$f(x) = o(g(x)) \text{ as } x \to \infty$$

if **for every** constant $M > 0$, there exists a constant k such that whenever $x > k$:

$$|f(x)| < M|g(x)| \quad \text{i.e.} \quad \lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0$$

$\longrightarrow f(x) = o(g(x))$ (little-oh) if *eventually* (namely for $x > k$), f grows strictly slower than **any** multiple of g.

Similarly, to describe the behaviour of a function near some real number a (often $a = 0$):

Definition (II)

We say that,

$$f(x) = o(g(x)) \text{ as } x \to a$$

if and only if:

$$\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = 0$$

The intuition behind little-oh notation is that f is $o(g)$ if $g(x)$ grows strictly faster than $f(x)$ as x approaches 0.

1. Review

2. Exercises

For the upcoming exercise recall the following proposition from your lecture notes:

Proposition 1

The following two statements are equivalent:

1. If f is differentiable at x_0 with derivative $f'(x_0)$
2. As $h \to 0$, $f(x_0 + h) = f(x_0) + hf'(x_0) + o(h)$

Note that using big-O notation, 2. can be expressed as $f(x_0 + h) = f(x_0) + O(h)$.

Exercise 4.8

Use O and o notation to describe the behaviour of the following functions as x approaches the given values:

(ii) $f_2(x) = \sqrt{1 + x^2}$ as $x \to 0$

Goal: understand/describe the behaviour of $f_2(x)$ as x approximates 0.

1. Little-o notation

1. **Function Value at 0**: $f_2(0) = \sqrt{1 + 0^2} = 1$
2. **First Derivative at 0**:

$$f_2'(x) = \frac{1}{2}(1 + x^2)^{-1/2} \times 2x; \quad f_2'(0) = 0$$

$\longrightarrow \boldsymbol{f_2(x) = 1 + o(x)}$ as $\boldsymbol{x \to 0}$

- Note that $f_2'(0) = 0$, i.e., the rate of change of $f_2(x)$ at $x = 0$ is zero. This is consistent with the statement $f_2(x) = 1 + o(x)$, which implies that as x approaches 0, the deviation of $f_2(x)$ from 1 (= the deviation of $o(x)$ from 0) is smaller than the deviation of $g(x) = x$ from 0. In other words, in the immediate vicinity of $x = 0$, the function $f_2(x)$ changes slower than linearly.
- As $x \to 0$, whatever change happens in $f_2(x)$ from the value 1 is less than the change in x itself. That is, $o(x)$ goes faster to 0 than x.

In summary, "$f_2(x) = 1 + o(x)$ as $x \to 0$" reflects that as x gets closer and closer to 0, the function $f_2(x)$ gets closer and closer to 1, with its deviation, $o(x)$, from 1 in a vicinity of $x = 0$ growing slower than linearly; in other words with its deviation, $o(x)$, going to zero faster than x.

Note: $f_2(x) = 1 + o(x)$ as $x \to 0$ implies that I could also express $f_2(x)$ as $f_2(x) = 1 + O(x)$ as $x \to 0$...

1. $f_2(x) = 1 + o(x)$ as $x \to 0$
   - This indicates that as x approaches 0, the difference between $f_2(x)$ and 1 is strictly smaller than the difference between x and 0. In other words, the function $f_2(x) - 1$ goes to 0 faster than x does.
2. $f_2(x) = 1 + O(x)$ as $x \to 0$
   - This indicates that as x gets close to 0, the function $f_2(x)$ is close to 1, and any deviation from 1 is at most linear in magnitude with respect to x.

As you can see, it is much more precise and informative to say $f_2(x) = 1 + o(x)$.

Big-O notation

Binomial Theorem for Fractional Exponent

Let $\alpha = \frac{p}{q}$ be a rational number (p, q integers). Then:

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{(\alpha)(\alpha-1)}{2!}x^2 + \ldots = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$$

Therefore,

$$f_2(x) = \sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 \ldots$$

The first non-constant term in the expansion of $f_2(x)$ around $x = 0$ is proportional to x^2.

$$\boldsymbol{\longrightarrow f_2(x) = 1 + O(x^2)} \text{ as } \boldsymbol{x \to 0}$$

When you're close to 0, the behaviour of $f_2(x)$ differs from the constant function 1 by an amount that is at most proportional to x^2.
Of course we can also say:

$$f_2(x) = 1 + \frac{1}{2}x^2 + O(x^4)$$

$$f_2(x) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + O(x^6)$$

$$\vdots$$

$f_2(x) = 1 + O(x^2)$ is more informative than $f_2(x) = 1 + O(x)$

- Since x^2 grows slower than x near 0, meaning it gets closer to 0 faster than x as x approximates 0. This implies that $f_2(x)$ is even closer to 1 than what is suggested by the $O(x)$ notation. This function behaves like a parabola near 0, which is flatter than a line when close to 0.
- Note that: $O(x^4)$ implies $O(x^3)$, which implies $O(x^2)$ which implies $O(x)$. Similarly, $o(x^4)$ implies $o(x^3)$, which implies $o(x^2)$ which implies $o(x)$.

(iii) $f_3(x) = \frac{x}{x^2+1}$ as $x \to 3$

- $f_3(3+h) = 0.3 - 0.08h + o(h)$
  - $f_3(3) = 0.3$
  - As x deviates from 3 by a small amount h, the function's value deviates from 0.3 at a rate of 0.08 times h.
  - The term $o(h)$ represents error terms that become negligible compared to h as h approaches 0.
- $f_3(3+h) = 0.3 + O(h)$
  - $f_3(3) = 0.3$
  - The $O(h)$ notation suggests that the deviation of $f_3(3+h)$ from 0.3 is at most linear in h as h approaches 0.

However, this notation doesn't specify the exact behaviour or rate of this deviation, i.e., it does not specify the exact coefficient in front of h.

(iv) $f_4(x) = \frac{x-8}{(x+2)(2x-1)}$ as $x \to \frac{1}{2}$

Remember we're interested on the behaviour of $f_4(x)$ as x approaches $\frac{1}{2}$.
Thus consider:

$$f_4\left(\frac{1}{2} + h\right) = \frac{\frac{1}{2} + h - 8}{\left(\frac{1}{2} + h + 2\right)\left(2 \times (\frac{1}{2} + h) - 1\right)} =$$

$$= \underbrace{\frac{1}{2h}}_{\to\infty \text{ as } h\to 0} \times \underbrace{\frac{h - 7.5}{h + 2.5}}_{\to -3<\infty \text{ as } h\to 0}$$

Exercise 4.10

[Only attempt this question if you are already familiar with the expansion of $\sin(x)$ *about* $x = 0$.*]*

A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{x}\sin(x) & \text{if } x < 0, \\ a & \text{if } x = 0, \\ x\sin\left(\frac{1}{x}\right) & \text{if } x > 0. \end{cases}$$

Is there a value of a which ensures that f is a continuous function at $x = 0$?

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