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AS1056 - Chapter 4, Tutorial 2. 07-11-2024. Notes.

Exercise 2.8 Use O and o notation to describe the behaviour of the following functions as x approaches the given values:

(ii) $f_2(x) = \sqrt{1+x^2}$ as $x \to 0$

1. **Function Value at 0**: $f_2(0) = \sqrt{1+0^2} = 1$
2. **First Derivative at 0**:

$$f_2'(x) = \frac{1}{2}(1+x^2)^{-1/2} \times 2x; \quad f_2'(0) = 0$$

Then *Proposition 1* in the lecture notes tells us that since:

1. f_2 is differentiable at $x_0 = 0$ with derivative $f_2'(0)$.

then this is equivalent to saying that

2. as $h \to 0$, $f_2(0+h) = f_2(0) + hf_2'(0) + o(h)$

Hence,

$$f_2(h) = 1 + h \times 0 + o(h) = 1 + o(h) \text{ with } o(h) \to 0 \text{ faster than linearly.}$$

We can corroborate our result by applying the definition.

Proof. W.t.s. that $f_2(x) = 1 + o(x)$ as $x \to 0$, which is the same as saying $f_2(x) - 1 = o(x)$ as $x \to 0$.

The definition tells us that,

$$f_2(x) - 1 = o(x) \text{ as } x \to 0 \iff \lim_{x\to 0} \frac{f_2(x)-1}{x} = 0$$

Thus, we just need to show that $\lim_{x\to 0} \frac{f_2(x)-1}{x} = 0$:

$$\lim_{x\to 0} \frac{f_2(x)-1}{x} = \lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x} \underset{\substack{\uparrow \\ \text{by l'Hôpital}}}{==} \lim_{x\to 0} \frac{\frac{1}{2}(1+x^2)^{-1/2} \times 2x}{1} = 0$$

$\square$

Now we are also asked to describe the behaviour of $f_2(x)$ using big-oh notation. An easy solution, would be just to say $f_2(h) = 1 + O(h)$, since we know that $o(h) \underset{\substack{\uparrow \\ \text{“implies”}}}{\implies} O(h)$.

Nevertheless, we can be a bit more thorough by implementing something called the *Binomial Theorem for Fractional Exponent* (c.f. tutorial slides). This tells us that we can express:

$$f_2(x) = \sqrt{1 + x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 ... = 1 + O(x^2) \text{ as } x \to 0$$

since, the higher order terms x^4, x^6... become negligible with respect to x^2 in a vicinity of $x = 0$, meaning $O(x^2)$ sufficiently captures the behaviour of $f_2(x)$ near zero.

Note this is more informative than saying just $f_2(h) = 1 + O(h)$. Because:

- Saying $f_2(h) = 1 + O(h)$ means the deviation of $f_2(h)$ from 1 grows at most linearly in a vicinity of $h = 0$.
- Saying $f_2(h) = 1 + O(h^2)$ means the deviation of $f_2(h)$ from 1 grows at most quadratically in a vicinity of $h = 0$ (and remember that “quadratically“ is slower than “linearly“ for $|x| < 1/2$).

In other words, $O(h^2) \to 0$ faster than $O(h) \to 0$, meaning that $f_2(h)$ is actually closer to 1 (in the vicinity of $h = 0$) than what we would conclude by just saying $f_2(h) = 1 + O(h)$.

Again, we can also corroborate our result by applying the definition.

Proof.

W.t.s. $f_2(x) = 1 + O(x^2)$ as $x \to 0$, i.e., $f_2(x) - 1 = O(x^2)$ as $x \to 0$.

The definition tells us that,

$$f_2(x) - 1 = O(x^2) \text{ as } x \to 0 \iff \left| \frac{f_2(x) - 1}{x^2} \right| \leq M \text{ for sufficiently small } x.$$

Hence, if we look at how the RHS (right hand side) absolute value looks like when x gets arbitrarily close to 0:

$$\lim_{x \to 0} \frac{f_2(x) - 1}{x^2} = \lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{x^2} \underset{\substack{\uparrow \\ \text{by l'Hôpital}}}{=} \lim_{x \to 0} \frac{\frac{1}{2}(1 + x^2)^{-1/2} \times \cancel{2x}}{\cancel{2x}} = \frac{1}{2} \leq M,$$

for some constant M. ∎

(iv) $f_4(x) = \frac{x-8}{(x+2)(2x-1)}$ as $x \to \frac{1}{2}$ We can try to implement the same approach as before,

1. **Function Value at 0**: $f_4(0) = \frac{0.5-8}{(0.5+2)(2\times 0.5-1)} = \lim_{x \to 0} \frac{-7.5}{2.5 \times x} = -\infty$

however, it seems we cannot implement *Proposition 1* for this case. Nevertheless, remember we're interested on the behaviour of $f_4(x)$ as x approaches $\frac{1}{2}$. Thus let us write:

$$f_4\left(\frac{1}{2}+h\right) = \frac{\frac{1}{2}+h-8}{\left(\frac{1}{2}+h+2\right)\left(2\times(\frac{1}{2}+h)-1\right)} =$$

$$= \underbrace{\frac{1}{2h}}_{\to\infty \text{ as } h\to 0} \times \underbrace{\frac{h-7.5}{h+2.5}}_{\to -3<\infty \text{ as } h\to 0}$$

so, the growth of $f_4(x)$ is dominated by the $\frac{1}{2h}$ term as $h \to 0$ (i.e. as $x \to \frac{1}{2}$). Therefore, we can write $f_4\left(\frac{1}{2}+h\right) = O\left(\frac{1}{h}\right) = O(h^{-1})$ as $h \to 0$. Meaning that on a vicinity of $x = \frac{1}{2}$, $f_4(x)$ grows at most as quickly as h^{-1} in the vicinity of $h = 0$ (where h^{-1} is a function that tends to infinity as $h \to 0$).