AS1056 - Mathematics for Actuarial Science. Chapter 3, Tutorial 2.

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1. Review & Exercises

Exercise 3.4.

Prove that $(2x^3 - 1)/(x+1)^3$ converges to 2 as $x \to \infty$. Write your proof formally, using εs and Δs.

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Limit at Infinity

We say that,

$$f(x) \to L \text{ as } x \to \infty \text{ or } \lim_{x\to\infty} f(x) = L$$

if for any $\varepsilon > 0$, we can find some $\Delta = \Delta(\varepsilon) > 0$ such that whenever $x > \Delta$ we have that $L - \varepsilon < f(x) < L + \varepsilon$, i.e., $|f(x) - L| < \varepsilon$.

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Hence, we want to show that there $\exists$ some $\Delta(\varepsilon) > 0$ s.t. for any $\varepsilon > 0$:

$$\left| \frac{2x^3 - 1}{(x+1)^3} - 2 \right| < \varepsilon \text{ for } x > \Delta(\varepsilon).$$

Of course you can always calculate the limit with your favourite technique:

$$\lim_{x\to\infty} \frac{2x^3-1}{(x+1)^3} = \lim_{x\to\infty} \frac{2x^3-1}{x^3+3x^2+3x+1} \underset{\uparrow}{=}$$

divide up/down by x^3

$$= \lim_{x\to\infty} \frac{2 - \overbrace{\frac{1}{x^3}}^{\to 0}}{1 + \underbrace{\frac{3}{x}}_{\to 0} + \underbrace{\frac{3}{x^2}}_{\to 0} + \underbrace{\frac{1}{x^3}}_{\to 0}} = 2$$

Or using l'Hôpital's rule that tells you that if:

1. $f(x)$, $g(x)$ are differentiable,
2. $\frac{d}{dx}g(x) \neq 0$, and,
3. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$

then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}g(x)} = L$$

Where c and L are any real number or $\pm\infty$. Then:

$$\lim_{x\to\infty} \frac{2x^3-1}{(x+1)^3} = \lim_{x\to\infty} \frac{6x^2}{3(x+1)^2} = \lim_{x\to\infty} \frac{12x}{6(x+1)} = \lim_{x\to\infty} \frac{12}{6} = 2.$$

But calculating the limit with any of this techniques wouldn't be a foundational rigorous proof!!

- Indeed on the first calculation we're relying on the fact that $\lim_{x\to\infty} 1/x^n = 0$ and on the second calculation we're relying on l'Hôpital's rule.
- For the proof to be complete we would need to prove that $\lim_{x\to\infty} 1/x^n = 0,\ n \in \mathbb{N}_{>0}$ on the first case, and to prove l'Hôpital's rule on the second case (which might be quite complicated...).

Thus, we'll formally proove that $\lim_{x\to\infty} \frac{2x^3-1}{(x+1)^3} = 2$ by using the definition of limit as $x \to \infty$.

Exercise 3.6

Suppose $x = \cos\left(\frac{1}{2}t\right)$ and $y = 2\sin(t)$ for $t \in [0, \pi]$.

(i) Show that the function $y = f(x)$ can be written explicitly as $y = \pm 4x\sqrt{1 - x^2}$.

Trigonometric identities

- $2\cos^2(\theta) - 1 = \cos(2\theta)$ (double angle formulas for sine and cosine)
- $\cos^2(\theta) + \sin^2(\theta) = 1$ (Pythagorean formula for sines and cosines)

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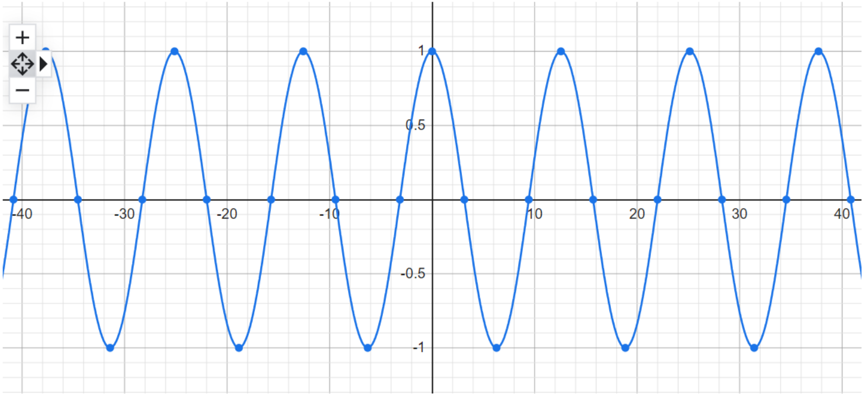
- $t \in [0, \pi]$
  - $x = \cos\left(\frac{1}{2}t\right)$
    - if $t = 0 \to x = \cos(0°) = 1$
    - if $t = \pi \to x = \cos(90°) = 0$
  - $y = 2\sin(t)$
    - if $t = 0 \to y = 2 \times \sin(0°) = 0$
    - if $t = \pi \to y = 2 \times \sin(180°) = 0$

Recall to check whether your calculator is radians or degrees (if in radians you'd put π if in degrees you'd put $180°$).

Graph for $\cos(0.5t)$



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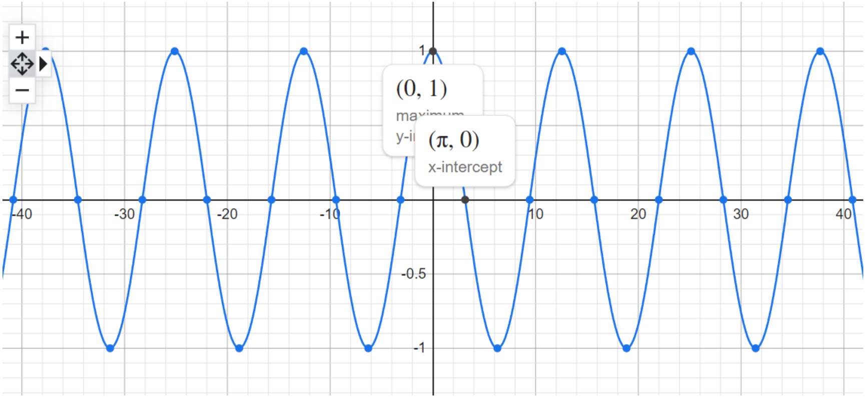
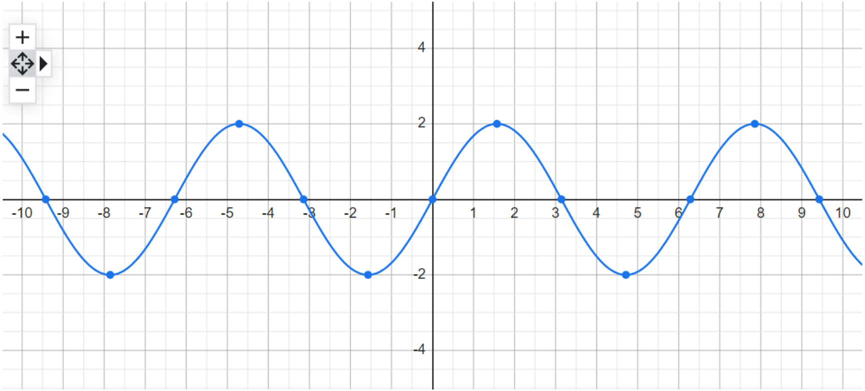


Graph for $2\sin(t)$



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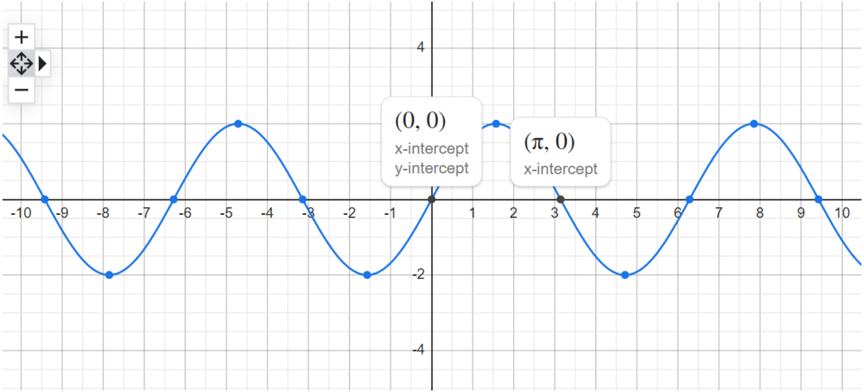

(iii) Work out the inverse function g such that $x = g(y)$.

Hint:

Consider the previously obtained expression $4x^4 - 4x^2 + \frac{1}{4}y^2 = 0$

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