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Hello everybody, let me provide some clarified "step by step" solutions to the exercise we discussed today. I know that many of you are unfamiliar with this "more theoretical" way of facing exercises, but what we want is that you kind of start "breaking the ice" with this stuff. So let's get to it.

Exercise 3.4 Prove that $(2x^3-1)/(x+1)^3$ converges to 2 as $x \to \infty$. Write your proof formally, using εs and Δs.

So this exercise is focused on proving that $\lim_{x\to\infty} f(x)$ equals 2 using the definition of the limit as $x \to \infty$. Let me emphasise that there's a significant distinction between merely calculating a limit and proving it. When we calculate a limit, we might use tools like L'Hôpital's rule. However, when asked to prove, we are delving deeper, seeking rigorous justification based on fundamental mathematical principles. Of course you could use L'Hôpital's rule to find the limit, but for a complete proof using that approach, you'd also need to prove L'Hôpital's rule itself (which might be more difficult than what we will do). So, for our purposes, it'll simpler and more direct to base our proof on the $\varepsilon - \Delta$ definition of a limit when $x \to \infty$.

Proof. Let me rewrite the limit as $x \to \infty$ definition in terms of our particular function and limit,

Definition:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^3-1}{(x+1)^3} = 2 \text{ if for any } \varepsilon > 0 \underset{\text{"exists"}}{\exists} \text{ a } \Delta = \Delta(\varepsilon) > 0 \text{ such that for } x > \Delta$$

the following holds:

$$2 - \varepsilon < \frac{2x^3-1}{(x+1)^3} < 2 + \varepsilon, \text{ i.e., } \left| \frac{2x^3-1}{(x+1)^3} - 2 \right| < \varepsilon.$$

We will prove $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^3-1}{(x+1)^3} = 2$ by showing it fits the above definition. Therefore, we need to find some $\Delta = \Delta(\varepsilon)$ (i.e. our value of Δ will depend of ε) such that, whenever $x > \Delta$ we have that $\left|\frac{2x^3-1}{(x+1)^3} - 2\right| < \varepsilon$ holds.

Deriving and inequality like $x > \Delta(\varepsilon)$ directly from $\left|\frac{2x^3-1}{(x+1)^3} - 2\right| < \varepsilon$ might be rather complicated. So, what we can start doing is to check whether we can bound $\left|\frac{2x^3-1}{(x+1)^3} - 2\right|$ by some nice and simple expression that afterwards allows us to get some inequality in terms of x and ε. That is, we will start by finding some simple expression, namely $g(x)$, such that $|f(x) - 2| < g(x)$. Then, since ε is an arbitrary positive number (that is, it can be whatever we want as long as it is > 0), we can just say $\varepsilon > g(x)$, which would imply $f(x) < g(x) < \varepsilon$. And then, we would use this inequality $\varepsilon > g(x)$ to solve for x, in order to get an inequality of the form $x > \Delta(\varepsilon)$.

So we start working with $\left|\frac{2x^3-1}{(x+1)^3} - 2\right|$ to see if we can bound it by some simple expression:

$$\left|\frac{2x^3-1}{(x+1)^3}-2\right| = \left|\frac{2x^3-1-2\times(x+1)^3}{(x+1)^3}\right| = \left|\frac{2x^3-1-2\times(x^3+3x^2+3x+1)}{(x+1)^3}\right| =$$

$$\left|\frac{2x^3-1-2x^3-6x^2-6x-2}{(x+1)^3}\right| = \left|\frac{-6x^2-6x-3}{(x+1)^3}\right| = \left|-\frac{6x^2+6x+3}{(x+1)^3}\right| \underset{\uparrow}{=}$$

by symmetry property of absolute values

$$= \left|\frac{6x^2+6x+3}{(x+1)^3}\right|$$

Note that we're interested in $x \to +\infty$, in particular in the values of x that are larger than some Δ (that we haven't figured out yet but that we know should be positive). So we're interested on the behaviour of x as it grows larger and larger, thus, the case where $x < 0$ is not relevant for our purposes. Hence, let us take $x > 0$; then we have,

$$\left|\frac{6x^2+6x+3)}{(x+1)^3}\right| = \frac{6x^2+6x+3}{(x+1)^3}$$

Since for $x > 0$ both the numerator and denominator on the above fraction will be positive, we can get rid of the absolute value. So, given that everything on the above expression is positive it is clear that the following is also true:

$$\left|\frac{6x^2+6x+3)}{(x+1)^3}\right| = \frac{6x^2+6x+3}{(x+1)^3} < \frac{6x^2+12x+6}{(x+1)^3} = \frac{6(x^2+2x+1)}{(x+1)^3} = \frac{6\cancel{(x+1)^2}}{(x+1)^{\cancel{3}}} =$$

$$= \frac{6}{x+1}$$

We thus have bounded $\left|\frac{6x^2+6x+3}{(x+1)^3}\right|$ by some other expression $\frac{6}{x+1}$ which indeed is quite simpler. Note that we could have bounded $\left|\frac{6x^2+6x+3}{(x+1)^3}\right|$ by many other expressions but the idea is always to get something simple. That is why we establish the bound as $\frac{6x^2+6x+3}{(x+1)^3} < \frac{6x^2+12x+6}{(x+1)^3}$, because replacing the 6 and the 3 of the left hand side by a 12 and a 6 on the right hand side, allows us to cancel things on the numerator/denominator and get to the simple expression $\frac{6}{x+1}$.

Thus, we have arrived to the conclusion that $\left|\frac{2x^3-1}{(x+1)^3}-2\right| = \left|\frac{6x^2+6x+3)}{(x+1)^3}\right| < \frac{6}{x+1}$ for $x > 0$. Therefore, we can now choose an arbitrary number $\varepsilon > 0$ and assume it is larger than $\frac{6}{x+1}$ (because ε can be whatever we want). Then we have:

$$0 \underset{\uparrow}{<} \left|\frac{2x^3-1}{(x+1)^3}-2\right| = \left|\frac{6x^2+6x+3}{(x+1)^3}\right| < \frac{6}{x+1} < \varepsilon$$

abs. values are always positive and $x > 0$

Finally, working on $\frac{6}{x+1} < \varepsilon$ we get $6 < \varepsilon(x+1)$, then, $x+1 > \frac{6}{\varepsilon}$ and $x > \frac{6}{\varepsilon} - 1$. Since we want that $\Delta(\varepsilon)$ works for any $\epsilon > 0$ and given that also $\Delta(\varepsilon) > 0$, let me define $\Delta(\varepsilon)$ as follows:

$$\Delta(\varepsilon) = \begin{cases} \frac{6}{\varepsilon} - 1, & \text{if } \varepsilon < 6 \\ 0 & \text{if } \varepsilon \geq 6 \end{cases}$$

So, for values of x such that $x > \Delta(\varepsilon) = \Delta$, we have that $\left|\frac{2x^3-1}{(x+1)^3}-2\right| = \left|\frac{6x^2+6x+3)}{(x+1)^3}\right| < \frac{6}{x+1} < \varepsilon$ holds. This is precisely the definition of the limit as $x \to \infty$. And this would end up the proof that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^3-1}{(x+1)^3} = 2$. □

Let us put some numbers on ε and Δ to see if we can further clarify what we have just achieved. Recall ε is just some arbitary stricly positive number, hence I can just tell you to assume $\varepsilon = 3$. In such case $\Delta = \Delta(\varepsilon = 3) = \frac{6}{3} - 1 = 1$. Given what we have just proved this implies that "for $x > \Delta = 1$, we have that $\left|\frac{2x^3-1}{(x+1)^3} - 2\right| < \varepsilon = 3$". Amazing! You can try plugging some values on your calculator to realise this indeed works.

Now, the idea on taking the limit as x goes to ∞ is that, given that the result we have proved holds for any $\varepsilon > 0$, I can just take ε to be as small as I want (as longs as it remains positive). For example, let's take $\varepsilon = 0.1$, then $\Delta = \Delta(\varepsilon = 0.1) = \frac{6}{0.1} - 1 = 59$, and hence, "for $x > \Delta = 59$, we have that $\left|\frac{2x^3-1}{(x+1)^3} - 2\right| < \varepsilon = 0.1$".

Now let ε tend to 0, then:

$$\Delta = \Delta\left(\lim_{\varepsilon\to 0} \varepsilon\right) = \lim_{\varepsilon\to 0} \frac{6}{\varepsilon} - 1 = \lim_{\varepsilon\to 0^+} \frac{6}{\varepsilon} - 1 = +\infty \quad \text{(the limit is from the right since } \varepsilon > 0\text{)}$$

Therefore, "for $x > \Delta = \infty$, in other words, for $x \to \infty$, we have that $\left|\frac{2x^3-1}{(x+1)^3} - 2\right| < \lim_{\varepsilon\to 0} \varepsilon$". That is, by making ε arbitrarily close to 0, we squeeze whatever is inside the absolute value, making $f(x) = \frac{2x^3-1}{(x+1)^3}$ arbitrarily close to 2. So, let $\varepsilon \to 0$, then as $x \to \infty$, $f(x) = \frac{2x^3-1}{(x+1)^3}$ converges to 2.

Try to little by little get the intuition...