Emilio Luis Sáenz Guillén
Doctoral Researcher in Actuarial Science

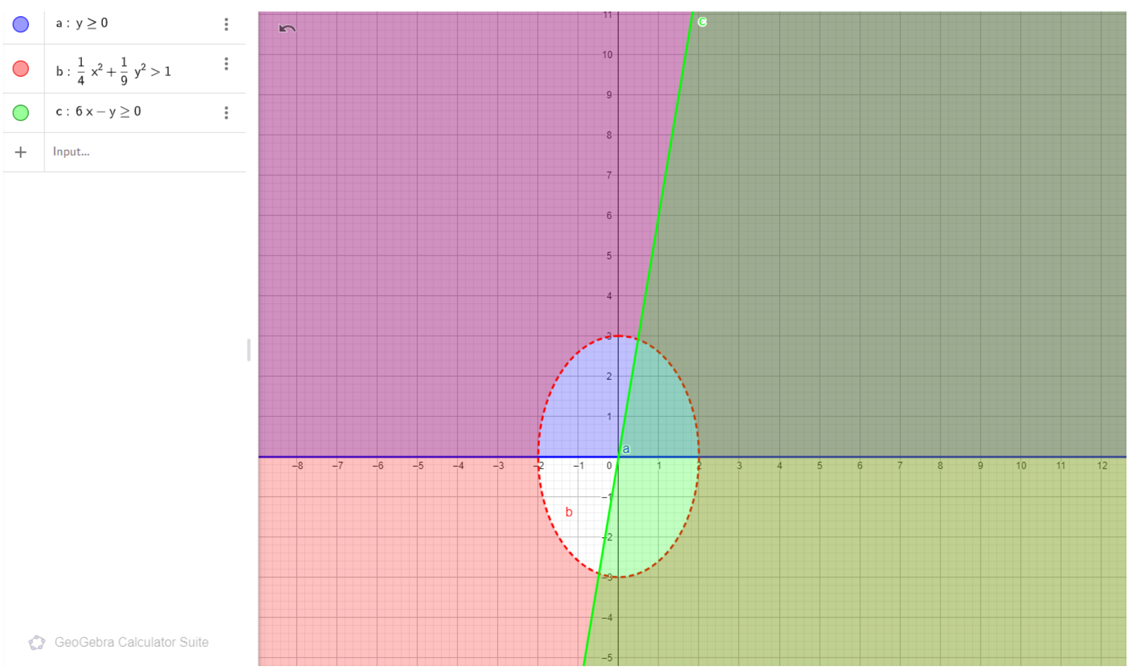
BAYES
BUSINESS SCHOOL
CITY ST GEORGE'S
UNIVERSITY OF LONDON

AS1056 - Chapter 2, Tutorial 2. 24-10-2024. Notes.

Exercise 2.7.

(i) Give a graphical illustration of the solution space of the inequalities

$$y \geq 0$$
$$\frac{1}{4}x^2 + \frac{1}{9}y^2 > 1$$
$$6x - y \geq 0$$



(ii) Suggest one further linear inequality which, when added to the others, would result in a solution space which has a finite, non-zero area.

The solution space obtained in (i) (up-right) is infinite. We need a line that closes it. In particular, we need a decreasing linear equation (i.e. negative slope) that lays in the 1st quadrant (top right of the plot) and that fulfils at least one of the following conditions:

– Intersects the x-axis at an x-value that is strictly larger than $x = 2$. In other words, a line that passes by $(x, 0)$ for $x > 2$ and by $(0, y)$, $y > 0$ (since I need this line to be decreasing). For example, consider the line that intersects the x axis at $(2.5, 0)$ and the y-axis at $(0, 1)$, the corresponding equation is $y = -\frac{1}{2.5} + 1$, and then a possible solution to the exercise would be $y \leq -\frac{1}{2.5} + 1$.

– Alternatively, we can consider a decreasing linear equation that lays strictly above the (top right) intersection point of $y = 6x$ and $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

Let us find out which is the point at which $y = 6x$ and $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ intersect. Replacing $y = 6x$ in the ellipsis equation:

$$\frac{1}{4}x^2 + \frac{1}{9}(6x)^2 = 1; \quad \frac{1}{4}x^2 + \frac{36}{9}x^2 = 1$$
$$\frac{1}{4}x^2 + 4x^2 = 1; \quad \frac{17}{4}x^2 = 1; \quad x^2 = \frac{4}{17}; \quad x = \pm\frac{2}{\sqrt{17}}$$

and,

$$y = 6 \times \left(\pm\frac{2}{\sqrt{17}}\right) = \pm\frac{12}{\sqrt{17}}$$

That is, alternatively to any decreasing line that intersects the x-axis at an x-value that is strictly larger than $x = 2$, we can consider any decreasing line that passes strictly above $(\frac{2}{\sqrt{17}}, \frac{12}{\sqrt{17}})$. In other words, a point that crosses $(x, \frac{12}{\sqrt{17}})$ for $x > \frac{2}{\sqrt{17}}$ and that crosses $(\frac{2}{\sqrt{17}}, y)$ for $y > \frac{12}{\sqrt{17}}$. For example, the line that passes by $(0, 4)$ and $(3, 0)$. To obtain the corresponding equation, consider the standard slope-intercept form of a linear equation: $y = mx + b$

$$\begin{cases} 0 = 3m + b \\ 4 = 0m + b; \quad b = 4 \end{cases}$$

$$\longrightarrow 3m = -4; \quad m = -\frac{4}{3}; \quad y = -\frac{4}{3}x + 4$$

Now, to have a non-empty solution space, we want our linear inequality to represent the area below this line, i.e., another possible solution for the exercise would be: $y \leq -\frac{4}{3}x + 4$.

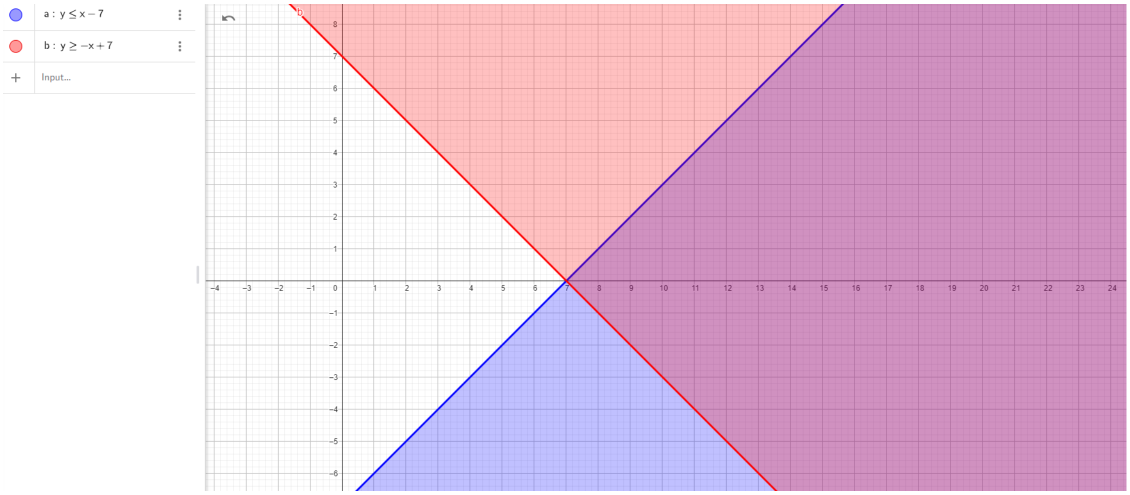
Exercise 2.11. Consider the simultaneous inequalities:

$$\begin{cases} x - |y| \geq 7 \\ y \geq A + x^2 \end{cases}$$

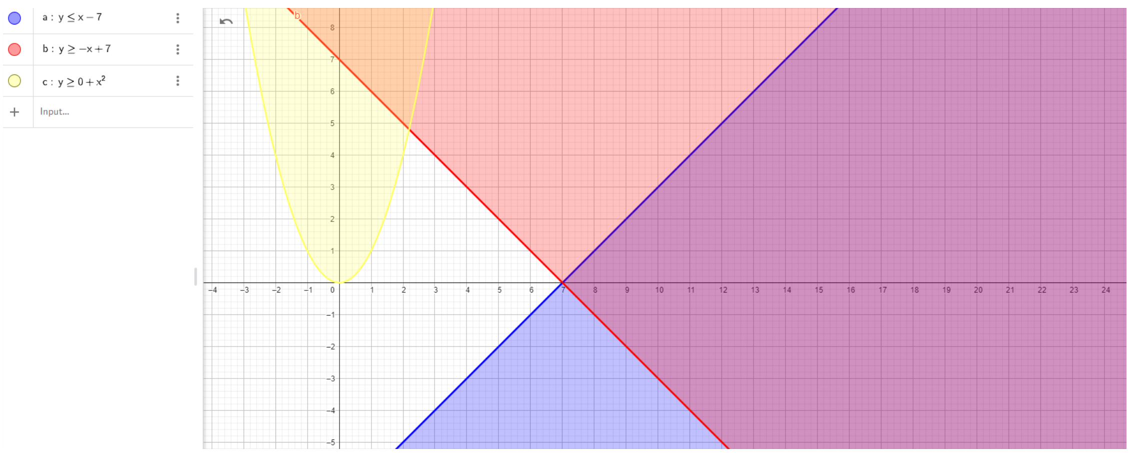
For which values of A is the solution space empty? **Hint:** It might be helpful to draw a diagram.

- $x - |y| \geq 7$

$$\begin{cases} x - y \geq 7 \text{ if } y \geq 0 \\ x + y \geq 7 \text{ if } y < 0 \end{cases} \implies \begin{cases} y \leq x - 7 \\ y \geq -x + 7 \end{cases}$$



- $y \geq A + x^2$. Let me plot the solution space of this inequality for $A = 0$:



For the two areas, respectively defined by the two inequalities above, to overlap we need to move the parabola downwards. In particular, the first point at which $x - |y| \geq 7$ and $y \geq A + x^2$ will meet is at the vertex defined by $x - |y| \geq 7$.

The vertex defined by $x - |y| \geq 7$ is just the point where $y = x - 7$ and $y = -x + 7$ intersect: $(7, 0)$. Hence, replacing in $y = A + x^2$ and solving for A:

$$0 = A + 7^2; \quad A = -49$$

In conclusion:

- For $A > -49$ the regions defined by each inequality do not intersect and hence the solution space is $\emptyset$.
- For $A \leq -49$ the two regions overlap and therefore the solution space is non-empty.

Exercise 2.9. Evaluate each of the following and give the limit as $K \to \infty$:
Assume $K > 0$. Let us start by calculating the following 4 integrals:

$$A: \int_0^K \lambda x e^{-\lambda x} dx = -Ke^{-\lambda K} + \frac{1}{\lambda}\left(1 - e^{-\lambda K}\right)$$
$$B: \int_0^K \lambda x e^{\lambda x} dx = Ke^{\lambda K} + \frac{1}{\lambda}\left(1 - e^{\lambda K}\right)$$
$$C: \int_{-K}^0 \lambda x e^{-\lambda x} dx = -Ke^{\lambda K} - \frac{1}{\lambda}\left(1 - e^{\lambda K}\right)$$
$$D: \int_{-K}^0 \lambda x e^{\lambda x} dx = Ke^{-\lambda K} - \frac{1}{\lambda}\left(1 - e^{-\lambda K}\right)$$

(iii) $\int_{-K}^K \lambda |x| e^{-\lambda x} dx$

$$\begin{aligned}
\int_{-K}^K \lambda |x| e^{-\lambda x} dx &= \underbrace{\int_0^K \lambda x e^{-\lambda x} dx}_{=A} - \underbrace{\int_{-K}^0 \lambda x e^{-\lambda x} dx}_{=C} = \\
&= -Ke^{-\lambda K} + \frac{1}{\lambda}\left(1 - e^{-\lambda K}\right) + Ke^{\lambda K} + \frac{1}{\lambda}\left(1 - e^{\lambda K}\right) = \\
&= e^{-\lambda K}\left(-K - \frac{1}{\lambda}\right) + \frac{1}{\lambda} + e^{\lambda K}\left(K - \frac{1}{\lambda}\right) + \frac{1}{\lambda} \\
&= \frac{2}{\lambda} + \underbrace{\underbrace{e^{\lambda K}}_{\to +\infty} \underbrace{\left(K - \frac{1}{\lambda}\right)}_{\to +\infty}}_{\to +\infty} - \underbrace{\underbrace{e^{-\lambda K}}_{\to 0} \underbrace{\left(K + \frac{1}{\lambda}\right)}_{\to +\infty}}_{\to 0} \longrightarrow +\infty \text{ as } K \to \infty.
\end{aligned}$$

Note: you can implement l'Hôpital's rule to corroborate that $\lim_{K\to\infty} e^{-\lambda K}\left(K + \frac{1}{\lambda}\right)$, however it is faster to just note that the exponential term $e^{-\lambda K}$ will go to 0 much faster (exponentially faster!) than the linear term $\left(K + \frac{1}{\lambda}\right)$ to $+\infty$.