

AS1056 - Mathematics for Actuarial Science. Chapter 18, Tutorial 1.

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Refreshing some concepts: Recurrence relations

A **recurrence relation** is an equation that expresses each element of a sequence $\{a_n\}$ as a function of the preceding ones.

$$a_n = f(n, a_{n-1}, a_{n-2}, \dots), \quad n \in \mathbb{N}^*$$

A recurrence relation is called a **difference equation** if it takes the form

$$a_n + b_1 a_{n-1} + \dots + b_k a_{n-k} = f(n), \quad n \geq k$$

for some constant coefficients b_1, \dots, b_k .

Solving Recurrence Relations other than Difference Equations

1. Linear Recurrence Relations with Variable Coefficients

$$a_n = b_n a_{n-1} + c_n \text{ for a predetermined sequence } \{b_n\}$$

In a case like this we proceed by multiplying on both sides by $\frac{1}{\prod_{i=1}^n b_i}$:

$$\underbrace{\frac{a_n}{b_n b_{n-1} \dots b_1}}_{=A_n} = \underbrace{\frac{b_n a_{n-1}}{b_n b_{n-1} \dots b_1}}_{=A_{n-1}} + \underbrace{\frac{c_n}{b_n b_{n-1} \dots b_1}}_{=C_n}$$

The solution of this first-order diff. equation is $A_N = A_0 + \sum_{n=1}^N C_n$.

2. Mathematical Induction

Formulate some induction hypothesis and then prove:

- (i) Base case
- (ii) Induction step

3. Convergence to a Limit

Exercise 18.7

Find an expression for the solution a_n of the recurrence relation

$$a_n = n + \frac{1}{n-1}a_{n-1}$$

with initial condition $a_1 = 1$.

Is this a difference equation?

Let us re-express it as:

$$a_n - \underbrace{\frac{1}{n-1}a_{n-1}}_{=b_n} = \underbrace{n}_{f(n)}$$

→ **No, since the coefficients are not constant, but depend of n .**

* *Linear recurrence relation with variable coefficients.*

Proof by induction

Mathematical induction is a method for proving that some statement $P(n)$ is true for any natural number n .

A **proof by induction** of some statement $P(n)$ consists of two steps:

- 1. Base Case:** Prove the statement $P(n)$ for an initial value of n , usually, $n = 0$ or $n = 1$. The purpose of this step is to show that the statement holds for the first number in the sequence of natural numbers.
- 2. Induction Step:** Assume that the statement $P(n)$ is true for some arbitrary natural number $n = k, n \geq 1$ (induction hypothesis). Finally, prove that, if $P(k)$ is true, then the next case, $P(k + 1)$, also holds.

Then, if the statement is true for one case (base case) + assuming that the statement holds for an arbitrary case k implies it holds for the next case $k + 1$, \implies the statement must be true for all $n \in \mathbb{N}$

Think of it as dominoes. The base case is like knocking over the first domino, and the inductive step is what ensures that if one domino falls, the next one will too. And, therefore, all the dominoes (i.e., the natural numbers) will fall.



Exercise 18.8

The sequence $\{a_n : n = 0, 1, 2, \dots\}$ has initial value $a_0 = 3$ and satisfies the equation $a_n = \frac{n}{n+1}a_{n-1} + 1$.

- (i) Work out the values of a_1, \dots, a_5 , expressing them as fractions rather than decimals.
- (ii) You should notice that the denominator of a_n is (in most cases) $n + 1$. Define $c_n = (n + 1)a_n$ and write down the values of c_n for n going from 0 to 6.
- (iii) Find A, B and C such that $c_n = A + Bn + Cn^2$.
- (iv) Formulate your findings as an inductive hypothesis.
- (v) Check that the hypothesis is satisfied for $n = 0$.
- (vi) Prove the inductive hypothesis.

} Proof by Induction

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