Emilio Luis Sáenz Guillén
Doctoral Researcher in Actuarial Science

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Exercise 18.7

A first-order linear recurrence relation with variable coefficients has the following generic form:

$$a_n = b_n a_{n-1} + c_n$$

where the coefficient b_n defines a sequence by itself. The exercise gives us:

$$a_n = \underbrace{\frac{1}{n-1}}_{=b_n} a_{n-1} + n$$

therefore,

$$b_n = \frac{1}{n-1} \quad \text{and} \quad \prod_{i=1}^{n} b_i = b_n b_{n-1}...b_1 = \frac{1}{(n-1)!}$$

Thus, in order to apply the "trick" presented in section 16.6.1 of the lecture notes, we multiply by $(n-1)!$ on both sides of our recurrence relation:

$$A_n := \frac{a_n}{b_n b_{n-1}...b_1} = (n-1)! a_n =$$
$$= \underbrace{\frac{(n-1)!}{n-1}}_{=(n-2)!} a_{n-1} + \underbrace{n(n-1)!}_{=n!} = \frac{b_n a_{n-1} + c_n}{b_n b_{n-1}...b_1} = A_{n-1} + C_n$$

Note that we have transformed our first-order linear recurrence relation with variable coefficients into a first-order difference equation $A_n = A_{n-1} + C_n$. Let us rewrite this as $A_n - A_{n-1} = C_n$ and notice that:

$$\begin{aligned}
n = 2 : & \quad \cancel{A_2} - A_1 = 2! \\
n = 3 : & \quad A_3 - \cancel{A_2} = 2! \\
\vdots & \quad \vdots \\
n = N-1 : & \quad \cancel{A_{N-1}} - A_{N-2} = (N-1)! \\
n = N : & \quad A_N - \cancel{A_{N-1}} = N!
\end{aligned}$$

Thus, taking the sum from $n = 2$ up to $n = N$ we get:

$$A_N - A_1 = \sum_{n=2}^{N} n!; \quad A_N = A_1 + \sum_{n=2}^{N} n!$$

Finally, applying the boundary condition $a_1 = 1$, we get:

$$A_1 = (1-1)!\underbrace{a_1}_{=1} = 1 \implies A_N = 1 + \sum_{n=2}^{N} n! \underset{\substack{\uparrow \\ 1! = 1}}{=} \sum_{n=1}^{N} n! = (N-1)! a_N$$

$$\longrightarrow a_N = \frac{1}{(N-1)!} \sum_{n=1}^{N} n!$$

Exercise 18.8 The sequence $\{a_n : \ n = 0, 1, 2, ...\}$ is defined by:

- $a_0 = 3$
- $a_n = \frac{n}{n+1} a_{n-1} + 1 \longrightarrow \underbrace{(n+1)a_n}_{=c_n} = \underbrace{na_{n-1}}_{=c_{n-1}} + (n+1)$

That is, defining $c_n = (n+1)a_n$, we can rewrite our recurrence relation as:

$$c_n = c_{n-1} + (n+1)$$

Note that again we have transformed a first-order linear recurrence relation with variable coefficients into a first-order difference equation. Indeed, without noticing it, we have implemented the same "trick" of section 16.6.1, since $\prod_{i=1}^{n} b_i = b_n b_{n-1} ... b_1 = \frac{n}{n+1} \frac{n-1}{n} ... \frac{1}{2} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$. Let us write some terms of this sequence:

First, using the definition of c_n,
$c_0 = (0+1)a_0 = 3$
Now, using the recurrence relation,
$c_1 = c_0 + (1+1)$
$c_2 = c_1 + (2+1) = c_0 + (1+1) + (2+1)$
$c_3 = c_2 + (3+1) = c_0 + (1+1) + (2+1) + (3+1)$
$\vdots$
$c_n = c_0 + (1+1) + (2+1) + (3+1) + ... + (n+1) = c_0 + \sum_{k=1}^{n}(k+1) = c_0 + \underbrace{\sum_{k=1}^{n} k}_{=\frac{n(n+1)}{2}} + \underbrace{\sum_{k=1}^{n} 1}_{=n}$

Then,

$$c_n = 3 + \frac{n(n+1)}{2} + n = \frac{6 + n^2 + n + 2n}{2} = \frac{n^2 + 3n + 6}{2}$$

that is,

$$a_n = \frac{n^2 + 3n + 6}{2(n+1)}$$

The exercise has already been solved. There is no need for mathematical induction or any further proof since the solution is entirely based on basic principles of algebra. Indeed, the inclusion of mathematical induction in the context of solving recurrence relations, serves more pedagogical purposes than practical ones.

Try to solve the exercise following the solutions provided by Russell, and consider why, after arriving at the induction hypothesis through such an approach, it is necessary to prove it.

$\sum_{k=1}^{n} \boldsymbol{n} = \frac{\boldsymbol{n(n+1)}}{\boldsymbol{2}}$
Proof:
Let $S = \sum_{k=1}^{k}$ and note:

$$S = 1 + 2 + ... + n$$
$$\underbrace{S = n + (n+1) + ... + 1}$$
$$2S = S + S = (n+1) + (n+1) + ... + (n+1) = n(n+1)$$
$$\longrightarrow S = \frac{n(n+1)}{2}$$

For the sake of completeness let me present the induction proof. Following the steps suggested by the solutions provided by Russell, you arrive to propose

$$a_n = \frac{n^2 + 3n + 6}{2(n+1)}$$

as induction hypothesis. The induction proof would then proceed as follows:

(v) **Base case.**

For $n = 0$, $a_0 = \frac{0^2+3\times 0+6}{2(0+1)} = \frac{6}{2} = 3$ ✓

(vi) **Induction step.** Assume that $a_n = \frac{n^2+3n+6}{2(n+1)}$ holds (induction hypothesis). Then, we want to show that:

$$a_{n+1} = \frac{(n+1)^2 + 3(n+1) + 6}{2(n+2)} = \frac{n^2 + 2n + 1 + 3n + 3 + 6}{2(n+2)} = \frac{n^2 + 5n + 10}{2(n+2)}$$

The recurrence relation we were given tells us that:

$$a_{n+1} = \frac{n+1}{n+2} a_n + 1 \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{=} \frac{\cancel{n+1}}{n+2} \frac{n^2 + 3n + 6}{2(\cancel{n+1})} + 1 = \frac{n^2 + 3n + 6 + 2n + 4}{2(n+1)} = \frac{n^2 + 5n + 10}{2(n+2)} \quad \checkmark$$

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