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AS1056 - Chapter 15, Tutorial 1. 19-03-2025. Notes.

Exercise 15.5

$$f(x,y) = \frac{(x-y)^2}{\sqrt{x^2+y^2}} \text{ over } A := \{(x,y) : x^2+y^2 < 1\}$$

We convert to polar coordinates by setting:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad dxdy = rdrd\theta$$

Let us simplify first the numerator,

$$(x-y)^2 = (r\cos(\theta) - r\sin(\theta))^2 = r^2(\cos(\theta)-\sin(\theta))^2 = r^2(\cos^2(\theta) - 2\cos(\theta)\sin(\theta) + \sin^2(\theta)) =$$
$$= r^2(\underbrace{\cos^2(\theta)+\sin^2(\theta)}_{=1 \text{ (by Pythagorean identity)}} - \underbrace{2\cos(\theta)\sin(\theta)}_{=\sin(2\theta) \text{ (by double angle identity)}}) = r^2(1-\sin(2\theta))$$

therefore,

$$\iint f(x,y)dxdy = \int_0^{2\pi}\int_0^1 \frac{r^2(1-\sin(2\theta))}{\cancel{r}}\cancel{r}drd\theta = \int_0^{2\pi}(1-\sin(2\theta)) \times \frac{1}{3} \times \underbrace{\left[r^3\right]_0^1}_{=1} d\theta =$$
$$= \frac{1}{3}\left[\theta + \frac{1}{2}\cos(2\theta)\right]_0^{2\pi} = \frac{1}{3}[2\pi + \frac{1}{2}\underbrace{\cos(4\pi)}_{=1} - 0 - \frac{1}{2}\underbrace{\cos(2\times 0)}_{=1}] = \frac{2}{3}\pi.$$

Exercise 15.9

$$A := \{(x, y) : \ \frac{1}{K} < x < K, \ \frac{1}{K} < y < K, \ xy < 1; \ K > 1\}$$

(i) First notice that the inequalities

$$\frac{1}{K} < x < K, \quad \frac{1}{K} < y < K, \quad K > 1$$

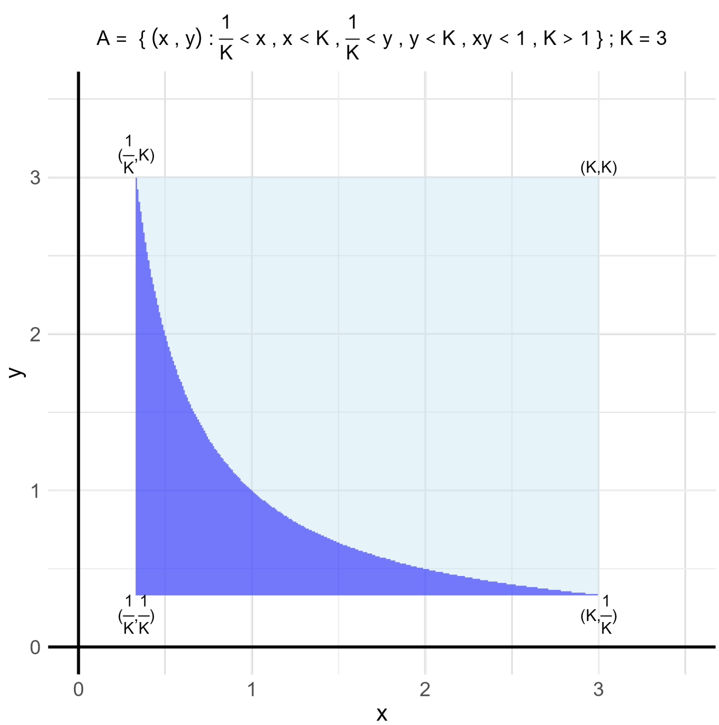
describe a square in the first quadrant with vertices: $(\frac{1}{K}, \frac{1}{K})$, $(\frac{1}{K}, K)$, (K, K), $(K, \frac{1}{K})$.

Second, the condition $xy < 1$ can be rewritten as $y < \frac{1}{x}$, meaning that A denotes the region *below* the hyperbola $y = \frac{1}{x}$.

Now, note that taking the limiting values of x and replacing in this hyperbola we get:

- if $x = \frac{1}{K}$ then $y = K$
- if $x = K$ then $y = \frac{1}{K}$

that is, the hyperbola passes by the second and third vertices of the square.



(ii) $\iint_A (xy)^c dxdy$ where $c \neq 1$

- Integral limits

$$\frac{1}{K} < x < K \text{ and } xy < 1, \text{i.e., } x < \frac{1}{y}$$

the lower bound of the inner integral is clear. For the upper bound we should choose the smaller between K and $\frac{1}{y}$, and we know that $\frac{1}{K} < y$, i.e., $\frac{1}{y} < K$.

– Inner integral

$$\iint_{\frac{1}{K}}^{\frac{1}{y}} (xy)^c dx = y^c \iint_{\frac{1}{K}}^{\frac{1}{y}} x^c dx = y^c \left[\frac{x^{c+1}}{c+1}\right]_{\frac{1}{K}}^{\frac{1}{y}} = \frac{y^c}{c+1}\left[\left(\frac{1}{y}\right)^{c+1} - \left(\frac{1}{K}\right)^{c+1}\right] = \frac{1}{c+1}\left[\frac{1}{y} - \frac{y^c}{K^{c+1}}\right]$$

– Outer integral

$$\frac{1}{c+1}\int_{\frac{1}{K}}^{K}\left(\frac{1}{y} - \frac{y^c}{K^{c+1}}\right)dy = \frac{1}{c+1}\left[\ln(y) - \frac{1}{K^{c+1}}\frac{1}{c+1}y^{c+1}\right]_{\frac{1}{K}}^{K} = \frac{1}{c+1}\left[\ln(y) - \frac{1}{c+1}\left(\frac{y}{K}\right)^{c+1}\right]_{\frac{1}{K}}^{K} =$$

$$= \frac{1}{c+1}\left[\ln(K) - \frac{1}{c+1}\left(\frac{\cancel{K}}{\cancel{K}}\right)^{c+1} - \ln\left(\frac{1}{K}\right) + \frac{1}{c+1}\left(\frac{1}{K^2}\right)\right] =$$

$$= \frac{1}{c+1}\left[2\ln(K)\underbrace{-\frac{1}{c+1} + \frac{1}{c+1}\left(\frac{1}{K^2}\right)}_{=-\frac{1}{c+1}\left((1-\frac{1}{K^{2c+2}}\right)}\right] = \frac{2\ln(K)}{c+1} - \frac{1}{(c+1)^2}\left(1 - \frac{1}{K^{2c+2}}\right)$$

(iii) $K \to \infty$

$$\lim_{K\to\infty}\frac{2\ln(K)}{c+1} - \frac{1}{(c+1)^2}\left(1 - \frac{1}{K^{2c+2}}\right) = \lim_{K\to\infty}\frac{2\ln(K)}{c+1} - \frac{1}{(c+1)^2}$$

The only way this can converge to a limit is if $c = K$:

$$\lim_{K\to\infty}\frac{2\ln(K)}{K+1} - \underbrace{\frac{1}{(K+1)^2}}_{\to 0} = \lim_{K\to\infty}\frac{2\ln(K)}{K+1} \underset{\substack{\uparrow \\ \text{L'Hôpital}}}{=} \lim_{K\to\infty}\frac{2(1/K)}{1} = 0$$