AS1056 - Mathematics for Actuarial Science. Chapter 12, Tutorial 1.

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19/02/2025

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Refreshing some concepts: Logic and Proof

The **contrapositive** of the statement $[P \implies Q]$ is $[\neg Q \implies \neg P]$.
In other words,

$$\left.\begin{array}{l}\text{'If } P \text{ is true then } Q \text{ is also true.'} \\ \text{'If } Q \text{ is false then it must be the case that } P \text{ is false.'}\end{array}\right\} \text{ are logically equivalent.}$$

Mathematical symbols:

- $\forall$ "for all"; $\exists$: "exists"
- $\neg$ "not" (logical negation)
- $\wedge$ "and"; $\vee$ "or"
- $\implies$ "implies"; $\iff$ "if and only if"
- $\therefore$ "therefore"

Exercise 12.11

We introduce the predicate Two(n), which is $\mathtt{TRUE}$ if and only if $n = 2$. Use this along with the predicates Even(n) and Prime(n) to express in logical notation the statement **"If n is a prime number and $n \neq 2$ then n is odd."** Write down the contrapositive of this statement in words and in logical notation.

- Two(n): "n is equal to two".
- Even(n): "n is equal to an even number".
- Prime(n): "n is equal to a prime number".

Exercise 12.7

Prove that, for every integer n, there exists an integer k such that

$$n^2 + n = 2k.$$

Hint. An integer n is called even if and only if there exists an integer k such that $n = 2k$.

Proof by induction

Mathematical induction is a method for proving that some statement $P(n)$ is true for any natural number n.

A **proof by induction** of some statement $P(n)$ consists of two steps:

1. **Base Case**: Prove the statement $P(n)$ for an initial value of n, usually, $n = 0$ or $n = 1$. The purpose of this step is to show that the statement holds for the first number in the sequence of natural numbers.
2. **Induction Step**: Assume that the statement $P(n)$ is true for some arbitrary natural number $n = k$, $n \geq 1$ (*induction hypothesis*). Finally, prove that, if $P(k)$ is true, then the next case, $P(k+1)$, also holds.

Then, if the statement is true for one case (base case) + assuming that the statement holds for an arbitrary case k **implies** it holds for the next case $k+1$: $\implies$ the statement must be true for all $n \in \mathbb{N}$

Think of it as dominoes. The base case is like knocking over the first domino, and the inductive step is what ensures that if one domino falls, the next one will too. And, therefore, all the dominoes (i.e., the natural numbers) will fall.

Exercise 12.13

This is a proof by mathematical induction of the statement

$S(n)$: In any group of n people, everyone has the same age.

1. The statement is true for $n = 1$.
2. **Assume it is true for n.** We need to demonstrate that it is true for $n + 1$.
3. Take a group of $n + 1$ people. Let us look at the group consisting of person 1, person 2, ..., person n. Because we are assuming that $S(n)$ is true, it follows that all of these people have the same age.
4. Now look at the group consisting of person 2, person 3, ..., person $n + 1$. This is a group of n people, so again it follows that they all have the same age.
5. Therefore person $n + 1$ and person 1 both have the same age as people 2, 3, ..., n — in other words, all $n + 1$ people have the same age.

Where does this proof **go wrong**?

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