Emilio Luis Sáenz Guillén
Doctoral Researcher in Actuarial Science

BAYES BUSINESS SCHOOL
CITY ST GEORGE'S
UNIVERSITY OF LONDON

AS1056 - Chapter 11, Tutorial 1. 12-02-2025. Notes.

Exercise 11.13

(i) Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; the eigenvalues of A satisfy the characteristic equation of A:

$$\det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda)) - bc = 0$$

operating,

$$ad - \lambda - \lambda d + \lambda^2 - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

We want to choose a, b, c and d in such a way that the solutions of this equation are $\lambda = 3$ and $\lambda = -1$. That is, remembering that equations can be expressed in factored form,

$$\lambda^2 - {\color{red}(a+d)}\lambda + {\color{blue}ad - bc} = \underbrace{\underbrace{(\lambda-3)(\lambda+1)}_{= \lambda^2 + \lambda - 3\lambda - 3}}_{= \lambda^2 - {\color{red}2}\lambda {\color{blue}-3}} = 0$$

Therefore, we need $a + d = 2$ and $ad - bc = 3$. For the sake of simplicity, let us just take $a = d = 1$. Then, we have $1 - bc = -3$, i.e., $bc = 4$; and again for the sake of simplicity, we just take $b = c = 2$. And finally we get that a posible solution for A is:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

(ii) Given the characteristic equation of A, $\lambda^2 - 2\lambda - 3 = 0$, we want to show that $A^2 - 2A - 3I = \mathbf{O}$. First, calculate A^2,

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

and then note that

$$A^2 - 2A - 3I = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Exercise 11.15

$$f(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \underbrace{\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad \theta \in [0, \pi]$$

Theory suggests:

- $\max_\theta f(\theta) = \max(\lambda_1, \lambda_2), \quad \theta \in [0, \pi]$
- $\min_\theta f(\theta) = \min(\lambda_1, \lambda_2), \quad \theta \in [0, \pi]$

where λ_1 and λ_2 denote the eigenvalues of matrix A.

Recall the following trigonometric identities,

- Pythagorean formula for sines and cosines:

  $\sin^2(\theta) + \cos^2(\theta) = 1$

- Double angle formulas for sine and cosine:

  $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
  $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

(i) We want to show that $\theta = \frac{\pi}{8}$ and that it is a maximum. Let us start by trying to simplify the quadratic form $f(\theta)$:

$$f(\theta) = \begin{pmatrix} 6\cos(\theta) + 2\sin(\theta) & 2\cos(\theta) + 2\sin(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} =$$

$$6\cos^2(\theta) + 2\sin(\theta)\cos(\theta) + 2\cos(\theta)\sin(\theta) + 2\sin^2(\theta) = 6\cos^2(\theta) + 2\sin^2(\theta) + 4\sin(\theta)\cos(\theta) =$$

$$= 4\cos^2(\theta) + 4\sin^2(\theta) + 2\cos^2(\theta) - 2\sin^2(\theta) + 4\sin(\theta)\cos(\theta) =$$

$$= 4\underbrace{[\cos^2(\theta) + \sin^2(\theta)]}_{=1} + 2\underbrace{[\cos^2(\theta) - \sin^2(\theta)]}_{=\cos(2\theta)} + \underbrace{4\sin(\theta)\cos(\theta)}_{2 \times \underbrace{2\sin(\theta)\cos(\theta)}_{= \sin(2\theta)}}$$

$$= 4 + 2\cos(2\theta) + 2\sin(2\theta)$$

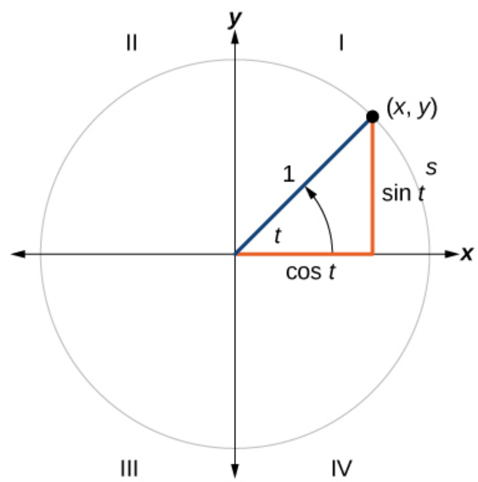
Taking the first derivative and equating to zero:

$$f'(\theta) = -4\sin(2\theta) + 4\cos(2\theta) = 0; \quad \not{4}\cos(2\theta) = \not{4}\sin(2\theta)$$

That is,

$$\cos(2\theta) = \sin(2\theta) \quad \text{or} \quad \frac{\sin(2\theta)}{\cos(2\theta)} = \tan(2\theta) = 1$$

Now, given the above identities, how much is 2θ? Maybe you remember by heart which angles $\theta \in [0, \pi]$ have a tangent equal to 1. In such case you can immediately deduce which are the turning points we're looking for. If you don't remember, you can note that $\cos(2\theta) = \sin(2\theta)$ implies that 2θ is an angle for which the values of sine and cosine are equal. Having in mind the unit circle,



it can come to us that —within the first period of these trigonometric functions[1]—, the angles at which sine=cosine are 45° ($=\frac{\pi}{4}$ radians) and 225° ($=\frac{5}{4}\pi$ radians; i.e., $45° + 180°$). Due to the periodic nature of sine/cosine functions there's an infinite number of solutions, i.e., an infinite number of angles for which the sine equals the cosine. In fact, the solution is any θ such that

$$2\theta = \frac{\pi}{4} + n\pi, \text{ i.e., } \theta = \frac{\pi}{8} + n\frac{\pi}{2}, \quad n \in \mathbb{Z}$$

Nevertheless, remind that we were told that $\theta \in [0, \pi]$, and thus the two only permissible solutions are indeed $\theta = \frac{\pi}{8}$ and $\theta = \frac{5}{8}\pi$.

Hence, we conclude that $\theta = \frac{\pi}{8}$ is a turning point. Now, let us show it is a maximum. First take the second derivative:

$$f''(\theta) = -8\cos(2\theta) - 8\sin(2\theta) = -8[\cos(2\theta) + \sin(2\theta)]$$

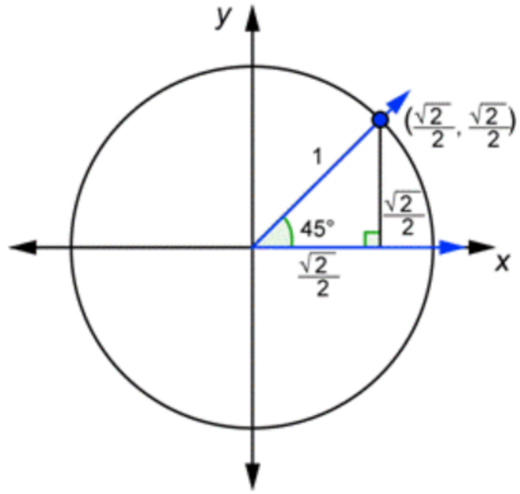
Evaluating the second derivative for $\theta = \frac{\pi}{8}$:

$$f''\left(\frac{\pi}{8}\right) = -8\left[\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right]$$

Since $\frac{\pi}{4}$ is an angle that lies in the first quadrant of the unit circle, it is clear that both its sine and cosine are positive (and if it is not clear to you, please have a look again to the unit circle image above). Hence we can already conclude that $f''\left(\frac{\pi}{8}\right) < 0$. Since $\theta = \frac{\pi}{8}$ is a turning point and $f''\left(\frac{\pi}{8}\right) < 0$, we conclude that $\theta = \frac{\pi}{8}$ is a maximum of f.

If you'd like to explicitly calculate what the value of $\cos(45°) = \sin(45°)$ is, let me show you a nice trick to do so without using a calculator. Consider an isosceles triangle (two equal sides and two equal angles) embedded on the unit circle and try to find it out by just using by Pythagoras theorem what $\cos(45°) = \sin(45°)$ should be. Note that the hypotenuse of such triangle will correspond to the radius of the unit circle:

[1]Period of the sine and cosine is 2π.



Therefore,

$$f''\left(\frac{\pi}{8}\right) = -8\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = -8\sqrt{2} < 0$$

(ii) To start with, let us work out:

$$f\left(\frac{\pi}{8}\right) = 4 + 2\underbrace{\cos\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}} + 2\underbrace{\sin\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}} = 4 + 2\sqrt{2}$$

On the other hand, the eigenvalues of A satisfy its characteristic equation:

$$\det(A - \lambda I) = \det\begin{pmatrix} 6-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = (6-\lambda)(2-\lambda) - 4 = 12 - 6\lambda - 2\lambda + \lambda^2 - 4 = \lambda^2 - 8\lambda + 8 = 0$$

$$\begin{aligned} \lambda &= \frac{8 \pm \sqrt{8^2 - 4 \times 1 \times 8}}{2 \times 1} \\ &= \begin{cases} \lambda_1 = 4 + \dfrac{\sqrt{32}}{2} = 4 + 2\sqrt{2} \\ \lambda_2 = 4 - \dfrac{\sqrt{32}}{2} = 4 - 2\sqrt{2} \end{cases} \end{aligned}$$

Thus, we confirm that $\max_{\theta \in [0,\pi]} f(\theta) = f(\frac{\pi}{8}) = \max(\lambda_1, \lambda_2) = \lambda_1 = 4 + 2\sqrt{2}$.

(iii) We already know from (i) that $\theta = \frac{5}{8}\pi$ is a turning point, hence just check whether $\theta = \frac{5}{8}\pi$ is a minimum and whether $f(\frac{5}{8}\pi) = \min(\lambda_1, \lambda_2) = \lambda_2 = 4 - 2\sqrt{2}$.