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AS1056 - Chapter 10, Tutorial 1. 05-02-2025. Notes.

Exercise 11.7
Sub-items (i) and (ii) are solved together.

$$A\mathbf{v} = u$$

(a) First let us deduce the geometric effects of transformation A, by working on it a bit:

$$A_1 = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}; \quad A_1^2 = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -8 \\ 8 & 0 \end{pmatrix}$$

- $\det(A_1) = 2 \times 2 - (-2) \times 2 = 8$; $\det(A_1^2) = 0 \times 0 - (-8) \times 8 = 64$ Thus, based on the properties of the determinant we can already conclude that:
  - ∗ Area of the image of the unit square under mapping A_1 is 8. Under A_1^2 is 64.
  - ∗ The linear transformation A_1 preserves the orientation of the vertices of the unit square.
- We can also deduce the angle of rotation that the linear transformation A will induce; recall the rotation matrix (see 10.7.3) and equate it to A by taking some scaling factor λ that makes these two matrices equal, i.e., $\lambda A_\theta = A$:

$$\lambda \underbrace{\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}}_{A_\theta} = \underbrace{\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}}_{A} = 2\sqrt{2} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

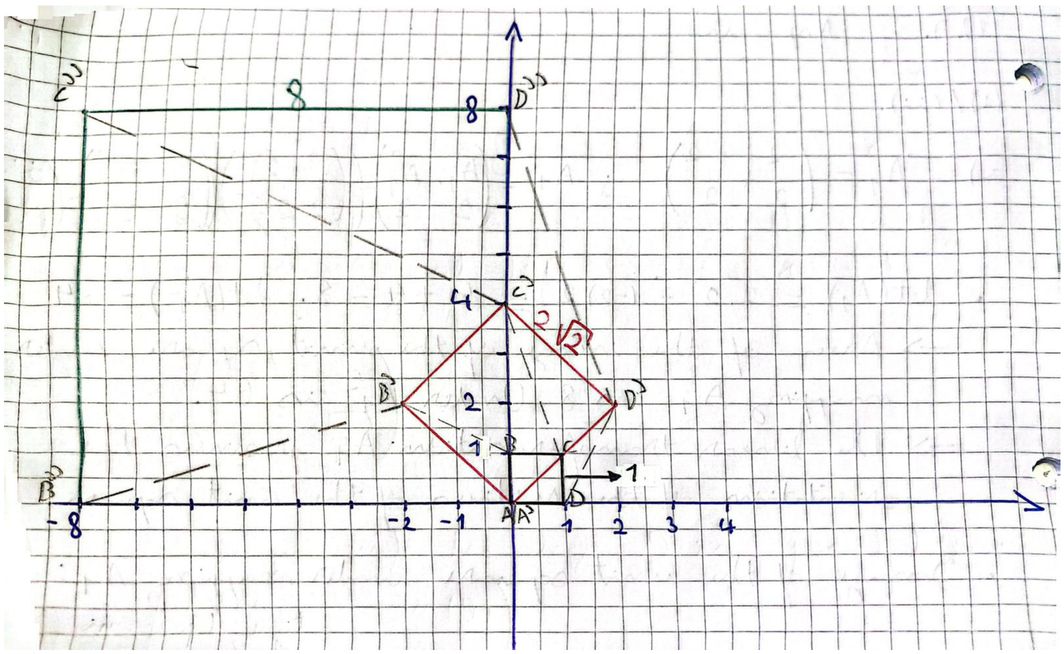
  in other words, the linear transformation A will rotate the unit square by 45°.
  Now, let us get back to what the exercise asked:
- Image of the unit square under mapping A_1:

$$A_1 \times \text{Unit Square} = \underbrace{\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}}_{A_1} \times \underbrace{\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}}_{\text{coordinates of the unit square}} = \begin{pmatrix} 0 & -2 & 0 & 2 \\ 0 & 2 & 4 & 2 \end{pmatrix} \begin{matrix} \rightarrow \text{x-coordinates} \\ \rightarrow \text{y-coordinates} \end{matrix}$$

$$A_1^2 \times \text{Unit Square} = \underbrace{\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}}_{A_1} \times \underbrace{\begin{pmatrix} 0 & -2 & 0 & 2 \\ 0 & 2 & 4 & 2 \end{pmatrix}}_{\text{coordinates of } A_1 \times \text{Unit Square}} = \begin{pmatrix} 0 & -8 & -8 & 0 \\ 0 & 0 & 8 & 8 \end{pmatrix}$$

  We have already the coordinates of the transformed squares, now let's plot them. To make it easier to follow where each vertex ends up once it is transformed let me name the vertices as follows:
  - ∗ Unit Square: A= $(0,0)$, B= $(0,1)$, C= $(1,1)$, D= $(1,0)$

* Transformed Square': A'$= (0,0)$, B'$= (-2,2)$, C'$= (0,4)$, D'$= (2,2)$
* Transformed Square": A"$= (0,0)$, B"$= (-8,0)$, C"$= (-8,8)$, D"$= (0,8)$



Now let us check whether the properties of the determinant we mentioned at the beginning fit with our drawing:

- *Area of image of the unit square under A.* Note that, diagonal(Square')$= 4$, hence using Pythagoras theorem, $4^2 = c^2 + c^2 = 2c^2$; $\frac{16}{2} = c^2$; $c = \sqrt{8} = 2\sqrt{2}$. And thus Area(Square$'$)$=(2\sqrt{2})^2 = 8$, which is equal to $|\det(A)|$.
- *Area of image of the unit square under A^2.* Area(Square$''$)$=8^2 = 64$, which is equal to $|\det(A^2)|$.

Each application of the mapping A_1 rotates the unit square by 45° and enlarges it by a factor of $2\sqrt{2}$ in each dimension, which in terms of the area corresponds to a factor of 8 at each transformation (Area unit square$= 1$, Area(Square$'$)$= 8 \times 1$, Area(Square$''$)$= 8 \times 8$). The clockwise orientation of the vertices of the initial unit square remains unchanged through the transformations.

Exercise 10.10

(i)

Area of the image of the unit square under $A = |\det(A)| = c \times c - 0 \times 0 = c^2$.

(ii)

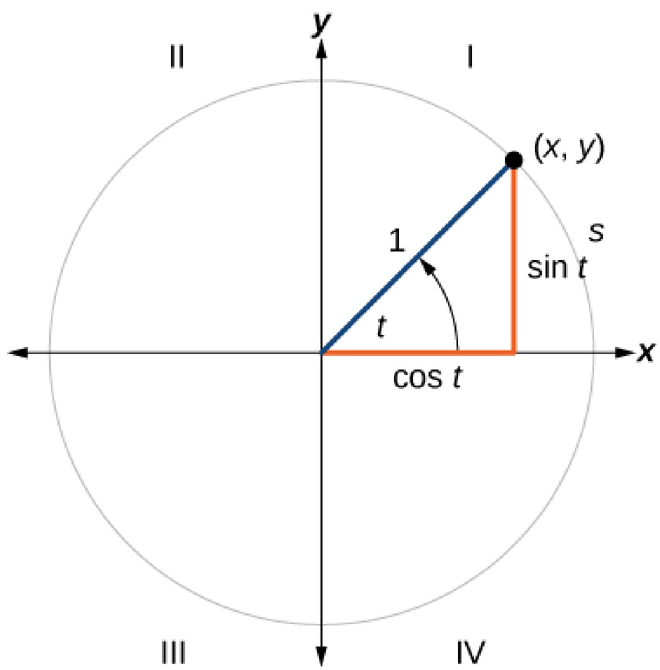
(a) Remember that the determinant of A gives us the scaling factor (in terms of the area) of the transformation. Thus what we want is $\det(A) = c^2 = 3$, i.e., $c = \sqrt{3}$. Then,

$$A = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$$

(b) Let us recall that, a rotation through the angle θ is represented by:

$$A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

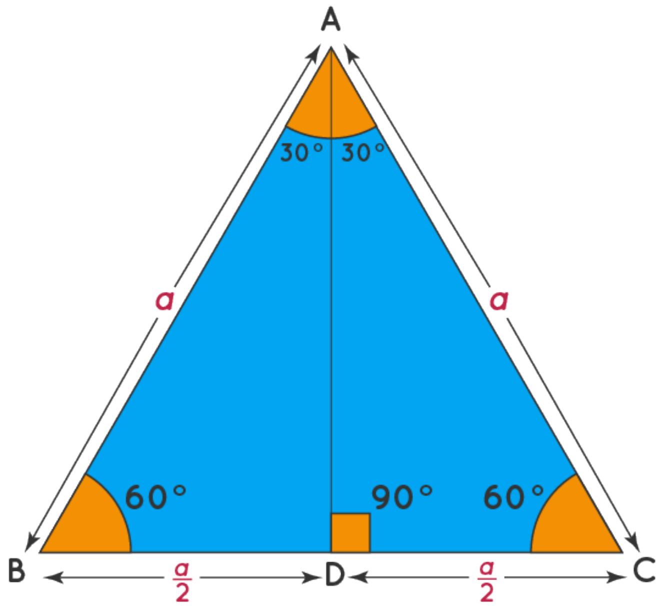
Now, when we apply the rotation matrix A_θ to any point in the plane, it rotates that point counterclockwise by the angle θ. It is kind of useful (and also to remind us what the sine and cosine are[1]) to think about this in terms of the following diagram:



In terms of the unit circle, you can think of θ as the amount by which you want to rotate the point (x, y) around the circle. After the rotation, the new coordinates will still be on the unit circle, just at a different angle.

Now we are given $\theta = \frac{\pi}{6}$ radians $= \frac{180}{6}$ degrees $= 30$ degrees. So, we just need to calculate $\cos(30°)$ and $\sin(30°)$. Maybe you remember by heart what the result of each of these is. Or maybe not. In such case, don't panic, and let me present an easy way of reminding it:

[1]The definitions of sine and cosine can be extended to any real value in terms of the lengths of certain line segments in a unit circle.



Consider an equilateral triangle. This type of triangles have equal sides and equal angles. Assume each side has length a. The sum of the angles of any triangle is equal to $180°$. Thus, each of the angles of an equilateral triangle has $60°$. If we split our equilateral triangle into two equally sized triangles, we get two $30° - 60° - 90°$ triangles. Therefore, and keeping in mind the above unit circle diagram (where a would equal 1), we can see that $\sin(30°) = \frac{a}{2} = \frac{1}{2}$, and, using Pythagoras, we can deduce that $\cos(30°) = \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$. Therefore the matrix we were looking for is:

$$A_\theta = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(c)

$$\underbrace{A}_{\text{enlargement}} \times \underbrace{A_\theta}_{\text{rotation}} = A_\theta \times A \longrightarrow \text{multiplication is commutative for square matrices}$$

$$A \times A_\theta = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{pmatrix}$$

Exercise 10.15

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \det(A) = ad - bc = 0$$

(i) $(a \quad b) \neq (0 \quad 0)$ and w.t.s. that $(c \quad d)$ is multiple of $(a \quad b)$. In other words, we want to show there $\exists$ some constant $\lambda \in \mathbb{R}$ such that $(c \quad d) = \lambda(a \quad b) = (\lambda a \quad \lambda b)$.

We have that $ad = bc$, then,

* $c = \frac{d}{b} \times a = \lambda a$ if $b \neq 0$ (a can be whatever); i.e., if $b \neq 0$ there $\exists$ $\lambda = \frac{d}{b}$ s.t. $c = \lambda a$, and also $d = \lambda b$.
* $d = \frac{c}{a} \times b = \lambda b$ if $a \neq 0$ (b can be whatever); i.e., if $a \neq 0$ there $\exists$ $\lambda = \frac{c}{a}$ s.t. $d = \lambda b$, and also $c = \lambda a$.

$\square$

(ii) Proof is similar.