AS1056 - Mathematics for Actuarial Science. Chapter 1, Tutorial 2.

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1. Review

2. Exercises

Review

- The real number r is ***rational*** if there exist integers p and q with $q \neq 0$ such that $r = \frac{p}{q}$. A real number that is not rational is called ***irrational***.

- An ***even number*** is an integer of the form $n = 2k$, where k is an integer.

Proof by contradiction (*reductio ad impossibile*)

To prove a statement by contradiction, start by assuming the opposite of what you would like to prove (*assume for the sake of contradiction*). Then show that the consequences of this premise are impossible. This means that your original statement must be true.

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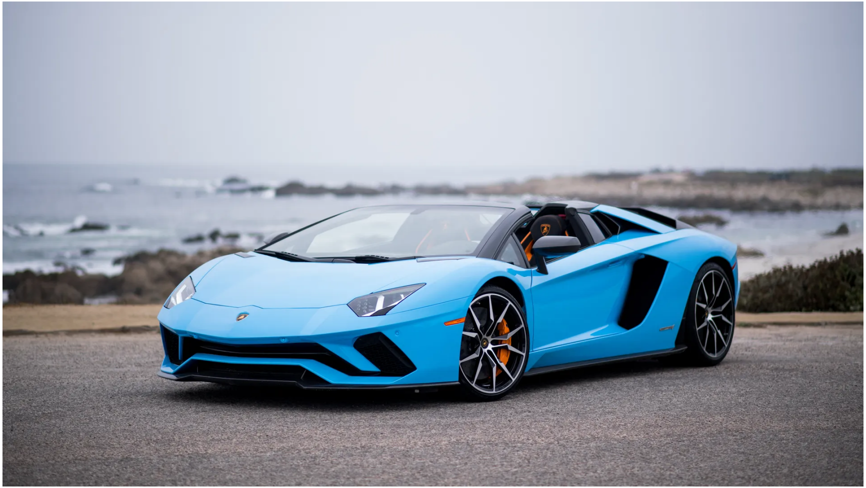
- Suppose you want to prove the statement: **"Not all cars are blue"**.
- Then, for a proof by contradiction, start assuming that **"all cars are blue"**. It is easily shown (literally) that there exist cars which are not blue, so the statement "all cars are blue" must be false.
- Therefore, you have derived a *contradiction* and so the original statement "not all cars are blue" must be true.

Without loss of generality, often abbreviated to WLOG, is a frequently used expression in math. The term is used to indicate that the following proof emphasizes on a particular case, but doesn't affect the validity of the proof in general.

Example: If three cars which are either red or blue, then there must be at least two cars of the same color.

A proof: Assume, without loss of generality, that the first car is red. If either of the other two cars is red, then we are finished; if not, then the other two cars must both be blue and we are still finished.



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Exercise 1.3: Prove that $\sqrt{2}$ is an irrational number.

Suppose $\sqrt{2} = \frac{p}{q}$, where $\frac{p}{q}$ is a rational number in its lowest terms.

Tasks:

(i) Show that p must be divisible by 2.

(ii) Show that q must be divisible by 2.

What can you conclude from the above results?

Hint: Consider squaring both sides of the equation and analyse the resulting equation for possible contradictions.

Exercise 1.6.

- $\mathbb{N}$ is countable (by definition).
- $\mathbb{Z}$ is countable.
- $\mathbb{R}$ is uncountable.

$\Big\}$ Check proofs in lecture notes. (applies to the $\mathbb{Z}$ and $\mathbb{R}$ items)

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} Check proofs in lecture notes.

(i) Prove that the union of two countable sets is countable.

(ii) Consider $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$, i.e., the irrational numbers.
If you take the infinitely many rational numbers away from the infinitely many real numbers, are you left with:

(a) the empty set,
(b) a countably infinite set or
(c) an uncountable infinite set?

Exercise 1.10. (ii)

Simplify

$$\sqrt{2}+\sqrt[4]{2}+\sqrt[8]{2}+\sqrt[16]{2}+\ldots$$

For **geometric series** remember the formula,

$$\boxed{\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}}$$

and in the case of an infinite geometric series:

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty}\sum_{k=0}^{n} x^k = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}, \text{ for } |x|<1$$

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