Emilio Luis Sáenz Guillén
Doctoral Researcher in Actuarial Science

AS1056 - Chapter 1, Tutorial 2. 17-10-2024. Notes.

Exercise 1.3: Prove that $\sqrt{2}$ is an irrational number. Suppose $\sqrt{2} = \frac{p}{q}$, where $\frac{p}{q}$ is a rational number in its lowest terms:

(i) Show that p must be divisible by 2.

(ii) Show that q must be divisible by 2.

What can you conclude from the above results?

Proof. Assume for the sake of contradiction that $\sqrt{2}$ is a rational number. Without loss of generality, let us also assume that it is a fraction in its lowest terms (simplest form),
i.e., $\underbrace{\sqrt{2} = \frac{p}{q} \quad p, q \in \mathbb{Z}, q \neq 0 \text{ and } \operatorname{hcf}(p, q) = 1.}$

if under this assumption we arrive to a contradiction, then we can conclude that $\sqrt{2}$ cannot be rational, i.e., $\sqrt{2}$ is irrational.

(i) $p = \sqrt{2} \times q; \quad p^2 = 2q^2 \implies p^2$ is even (and it is a perfect square, i.e., the square of an integer) $\implies$ p is even, since it cannot be odd because multiplication of odds always gives an odd number and we have just said that p^2 is even; in other words, the square root of an even perfect square is always even.
$\implies$ p is divisible by 2. ✓

(ii) $q = \frac{p}{\sqrt{2}}; \quad q^2 = \frac{p^2}{2} \underset{\uparrow}{=} \frac{(2k)^2}{2} = \frac{2^{\not{2}}k^2}{\not{2}} = 2k^2, \ k \in \mathbb{Z} \implies q^2$ is even (and a perfect square)
since p is even (by (i))
$\implies$ q is even, i.e., q is divisible by 2 ✓

Conclusion: p/q is not a fraction in its lowest terms (contradiction) $\implies$ $\sqrt{2}$ is irrational. □

Exercise 1.6

(i) Prove that the union of two countable sets is countable.

(ii) Consider $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$, i.e., the irrational numbers.
If you take the infinitely many rational numbers away from the infinitely many real numbers, are you left with:

(a) the empty set,
(b) a countably infinite set or
(c) an uncountable infinite set?

The intuition tells us that the correct answer is (c), however we need to prove it. That $\mathbb{I}$ is infinite is kind of evident (we can think about an infinity of numbers with decimals that cannot be expressed as a fraction); what is not so evident to show is that $\mathbb{I}$ is uncountable. Before starting the proof let me remind you that:

- $\mathbb{Z}$ is <u>countable</u>.
- $\mathbb{R}$ is <u>uncountable</u>.

} Check proofs in the lecture notes

Proof. Assume for the sake of contradiction that $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$ is countably infinite. By (i), we know that the union of two countably infinite sets, namely $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$ and $\mathbb{Q}$, should also be countably infinite (as the union of two countable sets is countable). However, $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q} \cup \mathbb{Q} = \mathbb{R}$ and we know that $\mathbb{R}$ is uncountably infinite, leading to a contradiction. Therefore, $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$ cannot be countable, meaning it must be the case that it is uncountably infinite. $\square$

Exercise 1.10

$$\sqrt{2} + \sqrt[4]{2} + \sqrt[8]{2} + \sqrt[16]{2} + ... = 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{8}} \times 2^{\frac{1}{16}} ... = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots} = 2^e = 2$$

since, applying the geometric series formula (see tutorial slides) to the exponent we get that,

$$e = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - \underbrace{\left(\frac{1}{2}\right)^0}_{=1} = \frac{1}{1-\frac{1}{2}} - 1 = 2 - 1 = 1$$