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AS1056 - Chapter 0, Tutorial 1. 08-10-2024. Notes.

Exercise A.6. A, B and C are three sets. Sets E and F are defined by

$$E = (A \cap B)\Delta C, \quad F = A \cap (B\Delta C).$$

(i) Show that $F \subset E$. First, let us work out E and F independently:

$$\cdot\, E = (A\cap B)\Delta C \overset{\text{symmetric difference}}{=} \underbrace{((A\cap B)\backslash C)}_{(A\cap B)\cap C^c} \cup \underbrace{(C\backslash (A\cap B))}_{C\cap(A\cap B)^c} \overset{\text{associativity of }\cap}{=} A\cap B\cap C^c \cup \underbrace{(A\cap B)^c}_{\substack{=(A^c\cup B^c)\\ \text{(by de Morgan's)}}} \cap C =$$

$$= A\cap B\cap C^c \cup (A^c\cup B^c)\cap C \overset{\text{distributivity of }\cap}{=} \underbrace{A\cap B\cap C^c}_{:=X} \cup \underbrace{A^c\cap C}_{:=Y} \cup \underbrace{B^c\cap C}_{:=Z} = X\cup Y\cup Z$$

$$\cdot\, F = A\cap(B\Delta C) \overset{\text{symmetric difference}}{=} A\cap((B\backslash C)\cup(C\backslash B)) = A\cap((B\cap C^c)\cup(C\cap B^c)) \underset{\text{distributivity/commutativity of }\cap}{=}$$

$$= \underbrace{A\cap B\cap C^c}_{:=X} \cup \underbrace{A\cap B^c\cap C}_{:=W} = X\cup W$$

Proof. Want to show (w.t.s.) that $F\subset E$, in other words, w.t.s. that, for any x, if $x\in F \implies x\in E$. There are two possible cases:

1. $x\in F$ and $x\in X$.
2. $x\in F$ and $x\in W$.

And looking at them separately, we can conclude as follows:

1. $x\in F$ and $x\in X \implies x\in E$, since $X\subset E$.
2. $x\in F$ and $x\in W = A\cap B^c\cap C$; clearly, $W = A\cap B^c\cap C \subset Z = B^c\cap C$ $\implies x\in Z \implies x\in E$.

□

(ii) Find an expression for $E\backslash F$ in terms of A, B and C.

$$E\backslash F = (\cancel{A\cap B\cap C^c}\cup (A^c\cup B^c)\cap C)\backslash(\cancel{A\cap B\cap C^c}\cup A\cap B^c\cap C)) =$$

$$= (A^c\cup B^c)\cap C\cap \underbrace{(A\cap B^c\cap C)^c}_{\substack{A^c\cup(B^c\cap C)^c = A^c\cup(B\cup C^c)\\ \text{(by de Morgan's)}}} = (A^c\cup B^c)\cap \underbrace{C\cap(A^c\cup B\cup C^c)}_{=C\cap(A^c\cup B)\ \text{(note that } C\cap C^c=\emptyset)} \overset{\text{commutativity of }\cap}{=}$$

$$= (A^c\cup B^c)\cap(A^c\cup B)\cap C = A^c\cap C = C\backslash A.$$

Exercise B.4. Write the function

$$f(x) = \frac{4x^4}{(2x-1)^2(x+1)}$$

as a linear term in x plus a remainder expressed in partial fractions.

$f(x)$ is a rational function, since it is a fraction of two polynomials:

$$f(x) = \frac{4x^4}{(2x-1)^2(x+1)} = \frac{p(x)}{q(x)}$$

and let us denote $m := \text{degree}(p(x)) = 4$ and $n := \text{degree}(q(x)) = 3$.

1. We can reduce a rational function to a polynomial (quotient of the division of $p(x)/q(x)$) plus a proper rational function (remainder of the division of $p(x)/q(x)$). The polynomial/quotient will be of degree $m-n$ if $m \geq n$ and of degree 0 if $m < n$.

   Let us find this polynomial/quotient by first operating on the denominator $q(x)$:

   $$q(x) = (2x-1)^2(x+1) = (4x^2-4x+1)(x+1) = 4x^3 + \cancel{4x^2} - \cancel{4x^2} - 4x + x + 1 =$$
   $$= 4x^3 - 3x + 1$$

   We know that the first term of the quotient of $p(x)/q(x)$ is just the higher degree term of the numerator divided by the higher degree term of the denominator: $4x^4/4x^3 = x$. Thus, we've already found the "linear <u>term</u> in x" that is hinted by the question statement. Now we need to find the remainder. Keep in mind:

   $$\frac{p(x)}{q(x)} = \underbrace{s(x)}_{\text{quotient}} + \frac{\overbrace{r(x)}^{\text{remainder}}}{q(x)}$$

   we know that $s(x) = x$ and based on this we can work out the remainder using the following trick:

   $$f(x) = x - x + \frac{4x^4}{(2x-1)^2(x+1)} =$$
   $$= x + \frac{1}{(2x-1)^2(x+1)}\{4x^4 - \underbrace{x(2x-1)^2(x+1)}_{\substack{x(x+1)(4x^2-4x+1)= \\ =(x^2+x)(4x^2-4x+1)= \\ =4x^4-\cancel{4x^3}+x^2+\cancel{4x^3}-4x^2+x= \\ =4x^4-3x^2+x}}\}$$
   $$= x + \frac{1}{(2x-1)^2(x+1)}\{\cancel{4x^4} - \cancel{4x^4} + 3x^2 - x\} = x + \underbrace{\frac{3x^2-x}{(2x-1)^2(x+1)}}_{\substack{\text{proper rational function} \\ \text{since degree num.} < \text{degree denom.}}}$$

   Note that above we have used the hint that the statement of the exercise implicitly gives us by saying that we need to write $f(x)$ as "linear term in x" plus something else. This meaning that we knew in advance that $s(x)$ was going to be just some coefficient times x, which we find by just dividing $4x^4/4x^3$. Without knowing this, we still know that any rational function can be expressed as a polynomial plus a proper rational function. In such case, to find this polynomial we can implement something called *polynomial long division*. The steps of polynomial long division are:

(i) Divide the highest-order term of the numerator by the highest-order term of the denominator, and put that in the answer.

(ii) Multiply the denominator by that answer, put that below the numerator. Subtract to create a new polynomial.

(iii) Repeat the process taking now the latter polynomial as the numerator.

(iv) Stop when the remainder is of a lower degree than the denominator or when it becomes zero.

$$
\begin{array}{r|l}
 & \quad x \\
\hline
4x^3-3x+1 \;) & 4x^4 \\
 & -(4x^4-3x^2+x) \\
\hline
 & \quad 0+3x^2-x
\end{array}
$$

and you see, that in this way we can also straightforwardly find the quotient $s(x)$ and the remainder $r(x)$.

2. We can express a proper rational function as a sum of simpler fractions through a process called *partial fraction decomposition.*

Recalling that for repeated roots we should include all its powers, the partial fractions form of the above proper rational functions is:

$$\frac{3x^2-x}{(2x-1)^2(x+1)} = \frac{A}{(x+1)} + \frac{B}{(2x-1)} + \frac{C}{(2x-1)^2}$$

$$3x^2 - x = A\underbrace{(2x-1)^2}_{=4x^2-4x+1} + B\underbrace{(2x-1)(x+1)}_{\substack{2x^2+2x-x-1= \\ =2x^2+x-1}} + C(x+1)$$

$$= A(4x^2-4x+1) + B(2x^2+x-1) + C(x+1) =$$

$$= (4A+2B)x^2 + (-4A+B+C)x + (A-B+C)$$

We get then the following system of equations:

$$\left\{\begin{array}{r} 3 = 4A + 2B \ (1) \\ \left.\begin{array}{r} -1 = -4A + B + C \\ 0 = A - B + C \ (3) \end{array}\right\} -1 = -5A + 2B \ (2) \end{array}\right.$$

Then, subtracting (1) − (2):

$$4 = 9A; \quad A = 4/9$$

Replacing $A = 4/9$ into (2):

$$-1 = -5 \times \frac{4}{9} + 2B; \quad 2B = -1 + \frac{20}{9} = \frac{11}{9}; \quad B = \frac{11}{18}$$

Finally, we derive C by replacing A and B on (3):

$$C = B - A = \frac{11}{18} - \frac{4}{9} = \frac{11}{18} - \frac{8}{18} = \frac{3}{18} = \frac{1}{9}$$

Thus, to conclude, $f(x)$ can be rewritten as:

$$f(x) = x + \frac{4/9}{(x+1)} + \frac{11/18}{(2x-1)} + \frac{1/6}{(2x-1)^2} = x + \frac{1}{18}\left[\frac{8}{x+1} + + \frac{11}{(2x-1)} + \frac{3}{(2x-1)^2}\right]$$

Exercise D.2 (i) The smallest integer which makes 3 appearances in Pascal's triangle is 6. Show that 6 makes exactly 3 appearances, i.e., that it cannot occur again lower down in the triangle.

We can find out (by hand, let's say) which are the 3 appearances of 6 in Pascal's triangle: .

$$^6C_1 = \binom{6}{1} = 6; \quad ^4C_2 = \binom{4}{2} = \frac{4!}{2!2!} = 6;$$
$$^6C_5 = \binom{6}{5} = 6 \quad \text{(note that } ^nC_1 = {}^nC_{n-1} = n)$$

Now, what might be more challenging is to show that 6 will not appear again in Pascal's triangle.

We can cover the entire Pascal's triangle through sets of the form $\left\{^nC_r = \binom{n}{r} : n = r, r+1, r+2, ...\right\}$, with $r \in \mathbb{N}^*$ (i.e., r being a natural number larger than zero: $1, 2, 3, ...$). Note that a specific integer can appear at most once in each of the latter sets. This is because the number in the position ^nC_r (for any r and any n) is derived from the sum of two numbers above it in the triangle, that is: $^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$. And hence these ^nC_r's grow as you move down the triangle, implying that $... {}^{n-1}C_r < {}^nC_r < {}^{n+1}C_r ...$.

- Consider the set $\left\{^nC_1 : n = 1, 2, 3, ...\right\}$, can we find an n such that $^nC_1 = 6$? Yes, as we have already mentioned, this is 6 since $^6C_1 = \binom{6}{1} = 6$.
- Consider the set $\left\{^nC_2 : n = 2, 3, 4, ...\right\}$, can we find an n such that $^nC_2 = 6$? Yes, as we have already mentioned, is 4 since $^4C_2 = \binom{4}{2} = 6$.
- Consider the set $\left\{^nC_3 : n = 3, 4, 5, ...\right\}$, can we find an n such that $^nC_3 = 6$? No since: $\binom{3}{3} = 1$, $\binom{4}{3} = 4$, $\binom{5}{3} = 10$,... and any other $\binom{k}{3}$ for $k > 5$ will be > 10 as $... {}^{n-1}C_r < {}^nC_r < {}^{n+1}C_r ...$
- Consider the set $\left\{^nC_4 : n = 4, 5, 6, ...\right\}$, can we find an n such that $^nC_4 = 6$? No since: $\binom{4}{4} = 1$, $\binom{5}{4} = 5$, $\binom{6}{4} = 15$,... and any other $\binom{k}{4}$ for $k > 6$ will be > 15 as $... {}^{n-1}C_r < {}^nC_r < {}^{n+1}C_r ...$
- Consider the set $\left\{^nC_5 : n = 5, 6, 7, ...\right\}$, can we find an n such that $^nC_5 = 6$? Yes, as we have already mentioned, this is 6 since $\binom{6}{5} = 6$.
- Consider the set $\left\{^nC_6 : n = 6, 7, 8, ...\right\}$, can we find an n such that $^nC_6 = 6$? No since: $\binom{6}{6} = 1$, $\binom{7}{6} = 7$, $\binom{8}{6} = 28$,... and any other $\binom{k}{6}$ for $k > 8$ will be > 28 as $... {}^{n-1}C_r < {}^nC_r < {}^{n+1}C_r ...$

And I hope that you can see that all the remaining sets ^nC_r for $r > 6$ will be formed by 1 and other values that are all larger than 6.