AS1056 - Mathematics for Actuarial Science. Chapter 9, Tutorial 2.

Emilio Luis Sáenz Guillén

Bayes Business School. City, University of London.

December 15, 2023

BAYES
BUSINESS SCHOOL
CITY, UNIVERSITY OF LONDON

Refreshing some concepts: Geometric series

$$S_n = \sum_{k=0}^{n-1} ar^k = \sum_{k=1}^{n} ar^{k-1} = \begin{cases} a\left(\frac{1-r^n}{1-r}\right), & \text{for } r \neq 1 \\ an, & \text{for } r = 1 \end{cases}$$

In particular for $n \to \infty$ we have that:

$$\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right) =$$

$$= \frac{a}{1-r}, \text{ for } |r| < 1$$

- A geometric series converges if and only if the absolute value of the common ratio is less than 1, i.e., $|r| < 1$.

Exercise 9.1

What is the radius of convergence, ρ, of the series $S_n = \sum_{i=1}^{n} (2x^2)^i$? What is the limit of the series when $|x| < \rho$?

For **geometric series** remember the formula,

$$\boxed{\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}}$$

the other formulas can be derived from it:

$$\sum_{k=1}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} - 1 = \frac{x - x^{n+1}}{1 - x}$$

$$\sum_{k=m}^{n} x^k = \sum_{k=0}^{n} x^k - \sum_{k=0}^{m-1} x^k = \frac{x^m - x^{n+1}}{1 - x}$$

Proof by induction

Mathematical induction is a method for proving that some statement $P(n)$ is true for any natural number n.
A **proof by induction** of some statement $P(n)$ consists of two steps:

1. **Base Case**: Prove the statement $P(n)$ for an initial value of n, usually, $n = 0$ or $n = 1$. The purpose of this step is to show that the statement holds for the first number in the sequence of natural numbers.
2. **Induction Step**: Assume that the statement $P(n)$ is true for some arbitrary natural number $n = k$ (induction hypothesis). Finally, prove that, if $P(k)$ is true, then the next case, $P(k+1)$, also holds.

If the statement is true for one case (the base case) + if we assume that the statement holds for an arbitrary case k implies it holds for the next case $k+1$:

$\implies$ Then the statement must be true for all natural numbers.

In simpler terms, think of it like dominoes. The base case is like knocking over the first domino, and the inductive step ensures that if one domino falls, the next one will too. Therefore, all the dominoes (or natural numbers) will fall.

Exercise 9.7

Let $f(x) = (2+x)^{-2}$.

(i) Show that the nth derivative of f is

$$f^{(n)}(x) = (-1)^n (n+1)!(2+x)^{-(n+2)}.$$

Refreshing some concepts: Ratio Rest

Proposition 8.1 (Ratio Test)

Given a series $S_n = \sum_{n=1}^{n} a_n$ and the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$$

The ratio test states that, the series S_n,

- converges absolutely —i.e. $\sum_{n=0}^{\infty} |a_n| = L$, for $L \in \mathbb{R}$—, if $\boldsymbol{r < 1}$;
- diverges, if $\boldsymbol{r > 1}$;
- if $r = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

Exercise 9.7

(ii) Evaluate $f^{(n)}(0)$ for each $n > 0$.

(iii) A series $S_n(x)$ is defined by

$$S_n(x) = \frac{1}{4} + \sum_{i=1}^{n} a_i x^i, \quad \text{where} \quad a_i = f^{(i)}(0).$$

What is the radius of convergence of the series S_n?