9.1.

$$S_n = \sum_{i=1}^{n} (-2x^2)^i$$

- The geometric series S_n converges as long as $|\underbrace{-2}_{<0}\underbrace{x^2}_{>0}| < 1 \Rightarrow 2x^2 < 1$; $x^2 < \frac{1}{2}$; $|x| < \sqrt{\frac{1}{2}}$

Therefore the radius of convergence of S_n is $\rho = \sqrt{\frac{1}{2}}$

- Take $|x| < \rho$, i.e., $|-2x^2| < 1$:

$$S_n = \sum_{i=1}^{n} (-2x^2)^i = \underbrace{\sum_{i=0}^{n} (-2x^2)^i}_{= \frac{1-r^{n+1}}{1-r}} - 1 =$$

$$= \frac{1-(-2x^2)^{n+1}}{1-(-2x^2)} - 1 = \frac{\cancel{1} - (-2x^2)^{n+1} - \cancel{1} + (-2x^2)}{1-(-2x^2)} =$$

$$= \frac{-2x^2 - (-2x^2)^{n+1}}{1-(-2x^2)} = -\frac{2x^2}{1+2x^2}\left(1-(-2x^2)^n\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\frac{2x^2}{1+2x^2}\left(1-\underbrace{(-2x^2)^n}_{\to 0}\right) = -\frac{2x^2}{1+2x^2}$$

9.7.

$$f(x) = (2+x)^{-2}$$

(i) W.t.s. that the nth derivative of f is:

$$f^{(n)}(x) = (-1)^n (n+1)! \, (2+x)^{-(n+2)}$$

Proof by induction:

1. Base case $(n=0)$

$$f^{(0)}(x) = f(x) = \underbrace{(-1)^0 (0+1)!}_{=1} \, (2+x)^{-(0+2)} = (2+x)^{-2} \checkmark$$

2. Induction step (prove that if the statement holds for one element $\Rightarrow$ it holds for the next element)

Assume that the formula $f^{(n)}(x)$ holds for some n (induction hypothesis). Then,

$$\underbrace{\frac{d}{dx}\left(f^{(n)}(x)\right)}_{=f^{(n+1)}(x)} = (-1)^n (n+1)! \, \frac{d}{dx}(2+x)^{-(n+2)} =$$

$$= (-1)^n (n+1)! \cdot \underbrace{(-(n+2))}_{=(-1)(n+2)} (2+x)^{-(n+3)} =$$

$$= (-1)^{n+1} (n+2)! \, (2+x)^{-(n+3)} \checkmark$$

And in this way (1.+2.), we've shown that the statement must be true $\forall n \in \mathbb{N}$. ▨

(ii) $$f^{(n)}(0) = (-1)^n (n+1)!\, \underbrace{(2+0)^{-(n+2)}}_{=2^{-n}\cdot 2^{-2}} =$$

$$= \frac{(n+1)!}{4} \cdot \left(-\frac{1}{2}\right)^n, \quad n > 0.$$

(iii)

$$S_n(x) = \frac{1}{4} + \sum_{i=1}^{n} a_i x^i, \quad a_i = f^{(i)}(0)$$

According to the ratio test the series $S_n(x)$ will be convergent if and only if:

$$\left| \frac{a_{n+1}\, x^{n+1}}{a_n \cdot x^n} \right| \longrightarrow r \text{ as } n \to \infty \text{ and } r < 1.$$

Thus, consider the above ratio:

$$\left| \frac{a_{n+1} \cdot x^{n+1}}{a_n \cdot x^n} \right| = \left| \frac{\frac{(n+2)!}{\cancel{4}} \cdot \left(-\frac{1}{2}\right)^{n+1} \cdot x^{n+1}}{\frac{(n+1)!}{\cancel{4}} \cdot \cancel{\left(-\frac{1}{2}\right)^n} \cdot \cancel{x^n}} \right| =$$

$$= \left| (n+2)\left(-\frac{1}{2}\right) \cdot x \right| \underset{\text{evenness property}}{=} \left| \frac{1}{2}(n+2) \cdot x \right| \underset{\text{multiplicativity property}}{=} \left| \frac{1}{2}(n+2) \right| \cdot |x| \underset{\text{by definition}}{=}$$

$$= \frac{1}{2}(n+2)\,|x|$$

Note that $\lim_{n\to\infty} \frac{1}{2}(n+2) = +\infty$. Therefore, and

given the mathematical convention $\infty \cdot 0 = 0$, the limiting ratio "r" will be greater than 1 unless $x = 0$.

Thus, the radius of convergence is $\rho = 0$, as the series diverges for all x except for $x = 0$.