AS1056 - Mathematics for Actuarial Science. Chapter 8, Tutorial 2.

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Exercise 8.1

(i) Write down the first 4 terms of the sequence $a_n = 2^{n-1}/(1+2^n)$.

(ii) What would you guess to be the limit, L?

(iii) In order to prove that L really is the limit, you need to come up with a way of choosing $n_0(\varepsilon)$ for every value of ε. How would you do this?

Limit of a sequence

We call L the limit of the sequence (a_n), which is written $a_n \to L$, or $\lim_{n\to\infty} a_n = L$, if the following condition holds:
For any real number $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon)$ such that, for every natural number $n \geq n_0(\varepsilon)$, we have $|a_n - L| < \varepsilon$

Exercise 8.7

A sequence is implicitly defined by the recursive equation $a_{n+1} = 16 + \frac{1}{2}a_n$ and has starting point $a_0 = 8$.

(i) Write down the values of a_n for $1 \leq n \leq 4$.

(ii) Identify the limit L of this sequence.

(iii) Define $b_n = a_n - L$. Write down an expression for b_n and, for $\varepsilon = 0.01$, find a value of n_0 such that $|b_n| < \varepsilon$ whenever $n \geq n_0$

Contractive sequences

A sequence a_n is called contractive if there exists $k \in [0, 1)$ such that

$$|a_{n+2} - a_{n+1}| \leq k|a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}$$

Theorem: Every contractive sequence is convergent.

Showing that the above sequence is contractive and using the above theorem, it is quite straightforward to calculate the limit of the sequence. However, we can also calculate the limit of the sequence without knowing what a contractive sequence is nor relying on any theorem. For such purposes, let me recall you the geometric series formulas:

$$s_n = \sum_{k=0}^{n-1} ar^k = \sum_{k=1}^{n} ar^{k-1} = \begin{cases} a\left(\frac{1-r^n}{1-r}\right), & \text{for } r \neq 1 \\ an, & \text{for } r = 1 \end{cases}$$

In particular for $n \to \infty$ we have that:

$$\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right) =$$

$$= \frac{a}{1-r}, \text{ for } |r| < 1$$

From last week...

Manipulating inequalities

Rule 1. Adding/subtracting the same quantity from both sides of an inequality leaves the inequality symbol unchanged.

Rule 2. Multiplying/dividing both sides by a positive number leaves the inequality symbol unchanged.

Rule 3. Multiplying/dividing both sides by a negative number reverses the inequality.

Rule 4. Applying any **monotonically increasing/decreasing** function to an inequality leaves the inequality symbol unchanged/ reverses the inequality. That is:

- $x \leq y \iff f(x) \leq f(y)$ if f is increasing.
- $x \leq y \iff f(x) \geq f(y)$ if f is decreasing.

In particular, raising both sides of an inequality to a power $n > 0$, when a and b are positive real numbers yields:

- $0 \leq a \leq b \iff 0 \leq a^n \leq b^n$
- $0 \leq a \leq b \iff a^{-n} \geq b^{-n} \geq 0.$

And raising both sides of an inequality to a power $n > 0$, when a and b are negative real numbers:

- $a \leq b \leq 0 \iff a^n \geq b^n \geq 0$
- $a \leq b \leq 0 \iff a^{-n} \leq b^{-n} \leq 0$

Last week we said: 'Squaring both sides of an inequality if both sides are positive/negative leaves the inequality symbol unchanged/reverses the inequality.'

Indeed, note the square function is increasing for positive values of x and decreasing for negative values of x.

For exercise 8.10 also consider using the following rule for definite integrals:

Domination rule of definite integrals

$$f(x) \geq g(x) \text{ on } [a, b] \implies \int_b^a f(x)dx \geq \int_b^a g(x)dx$$

Exercise 8.10

(i) Explain why if $n \in \mathbb{N}$ and $x \leq n$, with $x > 0$, then $n^{-\frac{3}{2}} \leq \int_{n-1}^{n} x^{-\frac{3}{2}} dx$ holds for $n \geq 2$.

(ii) Use part (i) to find a value U such that $\sum_{n=2}^{\infty} n^{-\frac{3}{2}} \leq U$

(iii) Show that $\sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty$.