8.1.

(i) $a_n = 2^{n-1}/(1+2^n)$

$a_1 = 2^0/(1+2^1) = \frac{1}{3}$

$a_2 = 2^1/(1+2^2) = \frac{2}{5}$

$a_3 = 2^2/(1+2^3) = \frac{4}{9}$

$a_4 = 2^3/(1+2^4) = \frac{8}{17}$

(ii) $$\lim_{n\to\infty} \frac{2^{n-1}}{1+2^n} \underset{\text{l'Hôpital}}{=} \lim_{n\to\infty} \frac{2^{n-1}\cdot \cancel{\ln(2)}}{2^n \cdot \cancel{\ln(2)}} = \frac{1}{2} = L$$

(iii)

Let us take a look at:

$$\left|a_n - \frac{1}{2}\right| = \left|\frac{2^{n-1}}{1+2^n} - \frac{1}{2}\right| = \left|\frac{\cancel{2^n} - 1 - \cancel{2^n}}{2(1+2^n)}\right| = \left|-\frac{1}{2(1+2^n)}\right| =$$

$$= \frac{1}{2(1+2^n)}$$

To show that $L = \frac{1}{2}$, from the definition of limit of a sequence, we need to show that for any $\varepsilon > 0$ we can find $n_0(\varepsilon) \in \mathbb{N}$ s.t. $\forall\, n \geq n_0(\varepsilon)$ $|a_n - L| < \varepsilon$ holds. That is,

$$\left|a_n - \frac{1}{2}\right| = \frac{1}{2(1+2^n)} < \varepsilon \;;\; \frac{1}{\varepsilon} < 2(1+2^n) = 2^{n+1} + 2$$

$$\frac{1}{\varepsilon} - 2 < 2^{n+1} \;;\; \ln\left(\frac{1}{\varepsilon} - 2\right) < (n+1)\ln(2)$$

$$\frac{1}{\ln(2)} \cdot \ln\left(\frac{1}{\varepsilon} - 2\right) - 1 < n$$

For instance, take:

$\varepsilon = 0.1 \Rightarrow \frac{1}{\ln(2)} \cdot \ln\left(\frac{1}{\varepsilon} - 2\right) - 1 = 2$ and so the best choice for $n_0(\varepsilon)$ is the smallest integer which is strictly greater than 2, i.e., $n_0(\varepsilon) = 3$.

In other words, $\left|a_n - \frac{1}{2}\right| < \varepsilon = 0.1$ holds for every $n \geq n_0(\varepsilon) = 3$.

8.7.

$$\underbrace{a_{n+1} = 16 + \frac{1}{2} a_n}\ ; \ a_0 = 8$$

(i) a_n for $1 \leq n \leq 4$

$$a_1 = 16 + \frac{1}{2} a_0 = 20$$

$$a_2 = 16 + \frac{1}{2} a_1 = 26$$

$$a_3 = 16 + \frac{1}{2} a_2 = 29$$

$$a_4 = 16 + \frac{1}{2} \cdot a_3 = 30.5$$

(ii) First, let us show that (a_n) is contractive. That is, show there exists $k \in [0,1)$ s.t. $|a_{n+2} - a_{n+1}| \leq k\,|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$.

$$|a_{n+2} - a_{n+1}| = \left|\cancel{16} + \frac{1}{2} a_{n+1} - \cancel{16} - \frac{1}{2} a_n\right| =$$

$$= \left|\frac{1}{2}(a_{n+1} - a_n)\right| = \frac{1}{2}\,\cancel{|a_{n+1} - a_n|}$$

$$\leq k\,\cancel{|a_{n+1} - a_n|} \quad ; \quad \frac{1}{2} \leq \underbrace{k < 1}$$

Note that every contractive sequence is convergent and thus we can be sure that L exists.

Taking the limit on both sides of our recursive equation:

$$\underbrace{\lim_{n\to\infty} a_{n+1}}_{=L} = 16 + \frac{1}{2} \underbrace{\lim_{n\to\infty} a_n}_{=L}$$

$L = 16 + \frac{1}{2} \cdot L \; ; \quad L = 32$

- $a_{n+1} = 16 + \frac{1}{2} a_n$

$a_0 = 8$

$a_1 = 16 + \frac{1}{2} \cdot a_0$

$a_2 = 16 + \frac{1}{2} \cdot \left(16 + \frac{1}{2} \cdot a_0\right) =$

$= 16 + 8 + \left(\frac{1}{2}\right)^2 \cdot a_0$

$a_3 = 16 + \frac{1}{2} \cdot \left(16 + 8 + \left(\frac{1}{2}\right)^2 \cdot a_0\right) =$

$= 16 + 8 + 4 + \left(\frac{1}{2}\right)^3 \cdot a_0 =$

$= \sum_{k=0}^{2} \frac{16}{2^k} + \left(\frac{1}{2}\right)^3 \cdot a_0$

$\vdots$

$a_n = \sum_{k=0}^{n-1} \frac{16}{2^k} + \left(\frac{1}{2}\right)^n \cdot a_0$

$\lim_{n\to\infty} a_n = \underbrace{\sum_{k=0}^{\infty} \frac{16}{2^k}} + \underbrace{\lim_{n\to\infty} \left(\frac{1}{2}\right)^n \cdot a_0}_{\to 0} = 32$

$= \frac{16}{1 - \frac{1}{2}}$

(iii)

$b_n = a_n - L$; $a_n = 32 + b_n$

$32 + b_{n+1} = a_{n+1} = 16 + \frac{1}{2} . a_n = 16 + \frac{1}{2}(32 + b_n)$

$\cancel{32} + b_{n+1} = \cancel{32} + \frac{1}{2} b_n$; $b_{n+1} = \frac{1}{2} b_n$

Similarly, we can write $b_n = \frac{1}{2} b_{n-1}$, then, $b_{n+1} = \frac{1}{2} \cdot \frac{1}{2} b_{n-1} = \left(\frac{1}{2}\right)^2 b_{n-1}$ and recursively we get, $b_{n+1} = \left(\frac{1}{2}\right)^{n+1} \cdot b_0$.

Now $b_0 = a_0 - L = 8 - 32 = -24$. Therefore, Therefore, $b_{n+1} = \left(\frac{1}{2}\right)^{n+1} \cdot (-24)$ and

$b_n = (-24) \cdot \left(\frac{1}{2}\right)^n$.

Finally, to prove that $L = 32$, from the definition of limit of a sequence, we need to show that for any $\varepsilon > 0$ we can find $n_0(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq n_0(\varepsilon)$, $|a_n - L| = |b_n| < \varepsilon$ holds.

So, we require,

$$|b_n| < \varepsilon \; ; \; \left|-24 \cdot \left(\frac{1}{2}\right)^n\right| < \varepsilon$$

$$24 \cdot \left(\frac{1}{2}\right)^n < \varepsilon \; ; \; \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{24}$$

$$n \cdot \underbrace{\ln\left(\frac{1}{2}\right)}_{=-\ln(2)} < \ln\left(\frac{\varepsilon}{24}\right) \; ; \; -n \ln(2) < \ln\left(\frac{\varepsilon}{24}\right)$$

$$n > -\frac{1}{\ln(2)} \cdot \ln\left(\frac{\varepsilon}{24}\right) \underset{\varepsilon = 0.01}{=} 11.22881869$$

Thus, we should set $n_0(\varepsilon) = 12$ when $\varepsilon = 0.01$.
In other words, $|b_n| < \varepsilon = 0.01$ holds for $n \geq n_0(\varepsilon) = 12$.

8.10

(i) $n \in \mathbb{N}$ and $x \leq n$; $x > 0$

$$x \leq n \implies x^{3/2} \leq n^{3/2} \; ; \; n^{-3/2} \leq x^{-3/2}$$

$\downarrow$

applying a monotonically increasing function on both sides of the inequality

By the domination rule of definite integrals

$$\int_{n-1}^{n} x^{-3/2}\, dx \geq \int_{n-1}^{n} n^{-3/2}\, dx = n^{3/2} \cdot \underbrace{\int_{n-1}^{n} dx}_{= \cancel{n} - \cancel{n} + 1} = n^{-3/2}$$

Why do we need $n \geq 2$?

$\longrightarrow x > 0 \implies n - 1 > 0 \; ; \; n > 1$

$\longrightarrow n \in \mathbb{N} \implies n \geq 2$