7.3.

$$\int_h^2 |x \ln(x)|\, dx \quad \text{for } 0<h<1$$

Note:

1. Since we're integrating from h to 2 and $0<h<1$, then $x \in (0, 2]$.

2. $x\ln(x) \geq 0$ if $x \geq 1$
   $x\ln(x) \leq 0$ if $0 < x \leq 1$

So we can divide the integral into two pieces:

$$\int_h^2 |x\ln(x)|\,dx = \int_h^1 -x\ln(x)\,dx + \int_1^2 x\ln(x)\,dx$$

by linearity

$\int x\ln(x)\,dx$? $\quad u = \ln(x);\ du = \frac{1}{x}\,dx$

$dv = x\,dx;\ v = \frac{x^2}{2}$

$$\int x\ln(x)\,dx = \ln(x)\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx =$$

$$= \ln(x)\cdot\frac{x^2}{2} - \frac{1}{2}\cdot\frac{x^2}{2} = \frac{x^2}{2}\cdot\left(\ln(x) - \frac{1}{2}\right)$$

$$\int_h^1 x\ln(x)\,dx = \frac{1^2}{2}\left(\cancel{\ln(1)} - \frac{1}{2}\right) - \frac{h^2}{2}\left(\ln(h) - \frac{1}{2}\right) =$$

$$= -\frac{1}{4} - \frac{h^2}{2}\left(\ln(h) - \frac{1}{2}\right)$$

$$\int_1^2 x\ln(x)\,dx = \frac{2^2}{2}\left(\ln(2) - \frac{1}{2}\right) - \frac{1^2}{2}\left(\cancel{\ln(1)} - \frac{1}{2}\right) =$$

$$= 2\ln(2) - 1 + \frac{1}{4} = 2\ln(2) - \frac{3}{4}$$

Hence,

$$\int_h^2 |x\ln(x)|\,dx = \frac{1}{4} + \frac{h^2}{2}\left(\ln(h) - \frac{1}{2}\right) + 2\ln(2) - \frac{3}{4} =$$

$$= 2\ln(2) - \frac{1}{2} + \frac{h^2}{2}\left(\ln(h) - \frac{1}{2}\right)$$

Does this converge to a limit as $h \to 0$?

Note,

$$\lim_{h\to 0} \frac{h^2}{2}\left[\ln(h) - \frac{1}{2}\right] = \lim_{h\to 0} \frac{\ln(h) - 0.5}{2 \cdot h^{-2}} =$$

$$= \lim_{h\to 0} \frac{1/h}{-4h^{-3}} = \lim_{h\to 0} -\frac{1}{\cancel{h}} \cdot \frac{h^3}{4} = \lim_{h\to 0} -\frac{1}{4}h^2 = 0$$

l'Hôpital

Thus,

$$\lim_{h\to 0} \int_h^2 |x\ln(x)\,dx| = 2\ln(2) - \frac{1}{2}$$

7.9. (ii)

- Line that intersects the x-axis at a value greater than 2, e.g: $(3, 0)$

- Line that intersects the y-axis at a value greater than 3, e.g: $(0, 4)$

Standard slope-intercept form of a linear equation:

$y = mx + b$

$$\begin{cases} 0 = 3m + b & \longrightarrow 3m = -4 \; ; \; m = -\frac{4}{3} \\ 4 = 0 \cdot m + b \; ; \; b = 4 \end{cases}$$

$$\longrightarrow y = -\frac{4}{3}x + 4$$

The inequality that expresses the area below the line is: $y < -\frac{4}{3}x + 4$

7.11. Assume $K > 0$:

$$A: \int_0^K \lambda . x e^{-\lambda x} dx = -K . e^{-\lambda K} + \frac{1}{\lambda}(1 - e^{-\lambda K})$$

$$B: \int_0^K \lambda . x . e^{\lambda x} dx = K . e^{\lambda K} + \frac{1}{\lambda}(1 - e^{\lambda K})$$

$$C: \int_{-K}^0 \lambda . x . e^{-\lambda x} dx = -K e^{\lambda K} - \frac{1}{\lambda}(1 - e^{\lambda K})$$

$$D: \int_{-K}^0 \lambda . x . e^{\lambda x} dx = K e^{-\lambda K} - \frac{1}{\lambda}(1 - e^{-\lambda K})$$

$$\text{(iii)} \int_{-K}^K \lambda |x| e^{-\lambda x} dx = \int_0^K \lambda x . e^{-\lambda x} dx - \int_{-K}^0 \lambda . x . e^{-\lambda x} dx =$$

$$= -K . e^{-\lambda K} + \frac{1}{\lambda}(1 - e^{-\lambda K}) + K e^{\lambda K} + \frac{1}{\lambda}(1 - e^{\lambda K}) =$$

$$= e^{-\lambda K} . \left(-K - \frac{1}{\lambda}\right) + e^{\lambda K}\left(K - \frac{1}{\lambda}\right) + \frac{2}{\lambda} =$$

$$= \frac{2}{\lambda} + \underbrace{e^{\lambda K}\left(K - \frac{1}{\lambda}\right)}_{\to +\infty} + \underbrace{\underbrace{e^{-\lambda K}}_{\to 0} \underbrace{\left(-K - \frac{1}{\lambda}\right)}_{\to -\infty}}_{\to 0} = +\infty$$

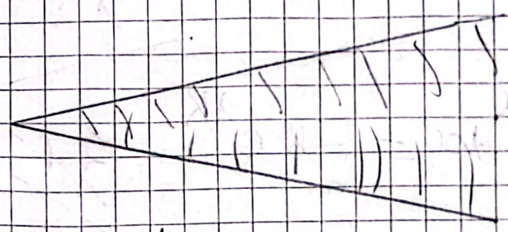
Note: you can also implement l'Hôpital to check that $\lim_{K \to \infty} e^{-\lambda K}\left(-K - \frac{1}{\lambda}\right) = 0$; however, it is much faster to just notice that the exponential term $e^{-\lambda K}$ will go to 0 much faster that the linear term $-K$ to $-\infty$.

7.13.

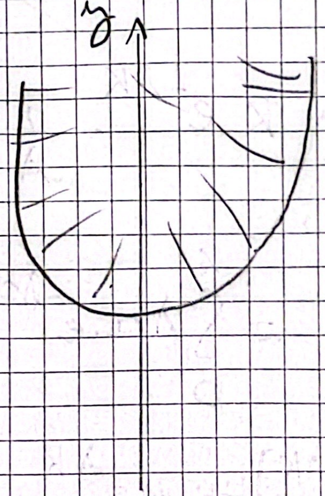
• $x - |y| \geq 7 \longrightarrow x - y \geq 7$ if $y \geq 0$

$x + y \geq 7$ if $y < 0$

$$\begin{cases} y \leq x - 7 \\ y \geq -x + 7 \end{cases}$$



• $y \geq A + x^2$



Note that (if $A=0$) the areas $y \geq x^2$ and $y \leq x-7$ do not overlap (if you're sceptic about this try to solve $x^2 = x - 7$ and you'll see this has no solution in $\mathbb{R}$). Therefore we need to move our parabola downwards, i.e., A needs to be negative. Note that moving downwards our parabola the first point at which $x - |y| \geq 7$ and $y \geq A + x^2$ will meet is at the vertex defined by $x - |y| \geq 7$.

• Vertex

$x - 7 = -x + 7$; $2x = 14$; $x = 7$

$y = 7 - 7 = 0$; Vertex $= (7, 0)$

And now we can calculate the first value of A for which an intersection of $y \geq A + x^2$ with $x - |y| \geq 7$ occurs:

$0 = A + 7^2$; $A = -49$

Thus,

- For $A > -49$ the curves do not intersect and the solution space is $\emptyset$.
- For $A \leq -49$ there is an overlap and therefore the solution space is non-empty.