AS1056 - Mathematics for Actuarial Science. Chapter 6, Tutorial 2.

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Refreshing some concepts[1]

- A function f(x) is called a rational function if it has the form:

$$f(x) = \frac{p(x)}{q(x)}$$

  for $p(x)$ and $q(x)$ being polynomial functions.
- $p(x)/q(x)$ is a proper rational function if the degree of p is less than the degree of q.

[1] Check 'Preliminary Materials', pp. 15-16.

- We can always reduce a rational function to a polynomial (quotient) plus a proper rational function (remainder). That is take the improper rational function $p(x)/q(x)$, then we can re-express it as:

$$\frac{p(x)}{q(x)} = \underbrace{s(x)}_{=\text{quotient}} + \frac{\overbrace{r(x)}^{\text{remainder}}}{q(x)}$$

where $\frac{r(x)}{q(x)}$ is now a proper rational function.

Partial Fractions Decomposition

We can express a proper rational function $p(x)/q(x)$ as a sum of simpler fractions.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
	• where $x^2 + bx + c$ cannot be factorised further	

✔ **Integration by partial fractions** is one popular method to integrate complex functions:

Step 1. Check whether the given integrand is a proper or improper rational function.

[2] Check exercise C.4 from the 'Preliminary Materials' for an example.

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Step 2. If the given function is an improper rational function perform 'polynomial long division'[2]. This will divide the function into a polynomial plus a proper rational function.

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Step 4. After decomposition, integrate each fraction separately.

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Exercise 6.8

Express $f(x) = \frac{(2-x)^2}{(2+x)^2(1+x)}$ in partial fractions. Hence find the definite integral of f from 3 to 4.

Integration of inverse functions

If we are looking for $\int_a^b f^{-1}(x)dx$, where f is continuous one-to-one function, we can make the substitution $x = f(y)$, implying that $dx = f'(y)dy$ and $y = f^{-1}(x)$ Then,

- The lower limit becomes $y = f^{-1}(a)$
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So, using integration by parts, we have,

$$\int_a^b f^{-1}(x)dx = \int_{f^{-1}(a)}^{f^{-1}(b)} yf'(y)dy = [yf(y)]_{f^{-1}(a)}^{f^{-1}(b)} - \int_{f^{-1}(a)}^{f^{-1}(b)} f(y)dy =$$

$$= bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(y)dy$$

Note that:

- 'f continuous' is a necessary and sufficient condition for Riemann integrability.
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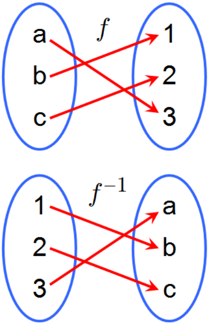
$\longrightarrow$ If f is a strictly monotone (i.e. strictly increasing/decreasing) continuous function then it has an inverse function f^{-1} which is also continuous (and is also strictly monotone; think why it makes all the sense it is like that!)



Exercise 6.6

The decreasing function $f(x) : \overbrace{\mathbb{R}^+ \cup \{0\}}^{=[0,+\infty)} \to (0, 1]$ is defined by

$$f(x) = \frac{1}{1 + \sqrt{x}}$$

We are looking for $\int_{0.25}^{0.5} f^{-1}(y)dy$.

(i) Calculate the inverse function $f^{-1}(y)$.

(ii) Integrate this function from 0.25 to 0.5.

(iii) Does this agree with the result you get from using the formula?