6.8.

- Express $f(x) = \dfrac{(2-x)^2}{(2+x)^2(1+x)}$ in partial fractions

- Find $\int_3^4 f(x)\,dx$

The generic form of the expression in partial fractions is:

$$f(x) = \frac{(2-x)^2}{(2+x)^2(1+x)} = \frac{A}{2+x} + \frac{B}{(2+x)^2} + \frac{C}{1+x}$$

$$(2-x)^2 = \frac{A\,(2+x)^{\cancel{2}}(1+x)}{\cancel{2+x}} + \frac{B\,\cancel{(2+x)^2}(1+x)}{\cancel{(2+x)^2}} + \frac{C\,(2+x)^2\cancel{(1+x)}}{\cancel{(1+x)}}$$

$$4-4x+x^2 = A\,(2+x)(1+x) + B\,(1+x) + C\,(2+x)^2$$

$$4-4x+x^2 = A\big(2+\overbrace{2x+x}^{=3x}+x^2\big) + B\,(1+x) +$$
$$+\,C\,(4+4x+x^2)$$

$$4-4x+x^2 = 2A+B+4C + (3A+B+4C)\,x +$$
$$+\,(A+C)\,x^2$$

So, we get the following system of equations:

$$\left.\begin{aligned} 2A+B+4C &= 4 \\ 3A+B+4C &= -4 \end{aligned}\right\}\quad A=-8$$

$$A+C = 1 \longrightarrow -8+C=1;\ C=9$$

$$2.(-8)+B+4.9=4\ ;\quad B = 4+16-36 = -16$$

$$f(x) = -\frac{8}{2+x} - \frac{16}{(2+x)^2} + \frac{9}{1+x}$$

Therefore,

$$\int_3^4 f(x)\,dx = -8\int_3^4 \frac{1}{2+x}dx - 16\int_3^4 \frac{1}{(2+x)^2}dx + 9\int_3^4 \frac{1}{1+x}dx =$$

$$= -8\left[\ln(2+x)\right]_3^4 - 16.\left[-\frac{1}{2+x}\right]_3^4 + 9.\left[\ln(1+x)\right]_3^4 =$$

$$= -8\left[\ln(6) - \ln(5)\right] + 16.\overbrace{\left[\frac{1}{6} - \frac{1}{5}\right]}^{=-1/30} + 9\left[\ln(5) - \ln(4)\right] =$$

$$= -8.\ln\left(\frac{6}{5}\right) - \frac{8}{15} + 9.\ln\left(\frac{5}{4}\right)$$

6.6.

$f(x): \mathbb{R}^+ \cup \{0\} \longrightarrow (0,1]$

$$f(x) = \frac{1}{1+\sqrt{x}} \; ; \; f'(x) = -\frac{1}{(1+\sqrt{x})^2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}} =$$

$$= -\frac{1}{2} \frac{1}{\underbrace{\underbrace{\sqrt{x}}_{>0} \underbrace{(1+\sqrt{x})^2}_{>0}}_{>0}} < 0 \quad \forall x \geq 0$$

$\longrightarrow f(x)$

$$\int_{0.25}^{0.5} f^{-1}(y)\,dy \; ?$$

(i) We want to solve for x the following expression:

$$y = \frac{1}{1+\sqrt{x}} \; ; \; 1+\sqrt{x} = \frac{1}{y} \; ; \; \sqrt{x} = \frac{1}{y} - 1$$

$$\sqrt{x} = \frac{1-y}{y} \; ; \; x = \left(\frac{1-y}{y}\right)^2 \quad \text{for } 0 < y \leq 1$$

i.e. $y \in (0,1]$

in other words: $f^{-1}(y) = \left(\frac{1-y}{y}\right)^2$.

(ii)

$$\int_{0.25}^{0.5} f^{-1}(y)\,dy = \int_{0.25}^{0.5} \left(\frac{1-y}{y}\right)^2 dy = \int_{0.25}^{0.5} \frac{1-2y+y^2}{y^2}\,dy =$$

$$= \int_{0.25}^{0.5} \frac{1}{y^2}\,dy - 2\int_{0.25}^{0.5} \frac{1}{y}\,dy + \int_{0.25}^{0.5} 1\,dy =$$

$$= \left[-\frac{1}{y}\right]_{0.25}^{0.5} - 2 \cdot \left[\ln(y)\right]_{0.25}^{0.5} + \left[y\right]_{0.25}^{0.5} =$$

$$= \left[4-2\right] - 2\left[\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{4}\right)\right] + \left[0.5 - 0.25\right] =$$

$$= 2 - 2\ln(2) + 0.25 = 2.25 - 2\ln(2)$$

(iii)

$$\int_{0.25}^{0.5} f^{-1}(y)\,dy = 0.5 \cdot f^{-1}(0.5) - 0.25 \cdot f^{-1}(0.25) - \int_{f^{-1}(0.25)}^{f^{-1}(0.5)} f(x)\,dx = *$$

$$f^{-1}(0.5) = \left(\frac{1-0.5}{0.5}\right)^2 = 1$$

$$f^{-1}(0.25) = \left(\frac{1-0.25}{0.25}\right)^2 = 9$$

$$* = \underbrace{0.5 \times 1 - 0.25 \times 9}_{=-1.75} - \int_1^9 \frac{1}{1+\sqrt{x}}\,dx$$

Substitute $u = \sqrt{x}$; $du = \frac{1}{2\sqrt{x}}\,dx$ and the limits of the integral become $g(x=9) = \sqrt{9} = 3$

$g(x=1) = \sqrt{1} = 1$

$$\int_1^9 \frac{1}{1+\sqrt{x}}\,dx = \int_1^3 \frac{1}{1+u}\,2u\,du = 2 \cdot \int_1^3 \left(1 - \frac{1}{1+u}\right)du =$$

$$= 2 \cdot \left[u - \ln(1+u)\right]_1^3 = 2\left[3 - \ln(4) - 1 + \ln(2)\right] =$$

$$= 2 \cdot \left[2 + \ln\left(\frac{1}{2}\right)\right] = 4 + 2\ln\left(\frac{1}{2}\right) = 4 - 2\ln(2)$$

$$\longrightarrow \int_{0.25}^{0.5} f^{-1}(y)\,dy = -1.75 + 4 - 2\ln(2) = \underline{2.25 - 2\ln(2)}$$