AS1056 - Mathematics for Actuarial Science. Chapter 5, Tutorial 2.

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Turning points/Local extrema: Maximums and Minimums

Definition: Local Maximum and Minimum

Suppose a function f is defined on some interval. For sufficiently small $\delta > 0$ and for all x contained in the interval $[b - \delta, b + \delta]$:

- If $f(b) \geq f(x)$, then at $x = b$ there is a **(local) maximum**.
- If $f(b) \leq f(x)$, then at $x = b$ there is a **(local) minimum**.

In plain words,

- f has a (local) maximum at $x = b$ if there exists an interval (a, c), $b \in (a, c)$, such that, for all $x \in (a, c)$, $f(b) \geq f(x)$.
- f has a (local) minimum at $x = b$ if there exists an interval (a, c), $b \in (a, c)$, such that, for all $x \in (a, c)$, $f(b) \leq f(x)$.

Exercise 5.3

If f is differentiable and b is a turning point, is it true that $f'(b) = 0$?

Hint:

Use the definition of maximum and the following proposition to arrive to a contradiction.

Proposition 2.1

The following two statements are equivalent:

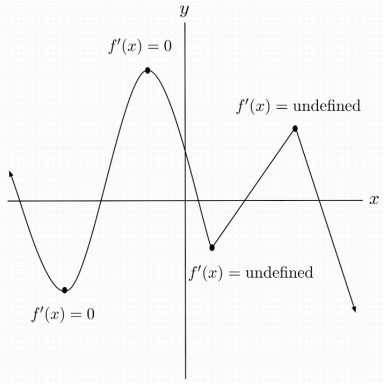
1. f is differentiable at x_0 with derivative $f'(x_0)$
2. As $h \to 0$, $f(x_0 + h) = f(x_0) + hf'(x_0) + o(h)$

Definition: Critical Point

The function f is said to have a critical point at x if:

$$f'(x) = 0 \quad \text{or} \quad f'(x) \text{ is undefined.}$$

These points can be classified as local minima, local maxima, or points of inflection.

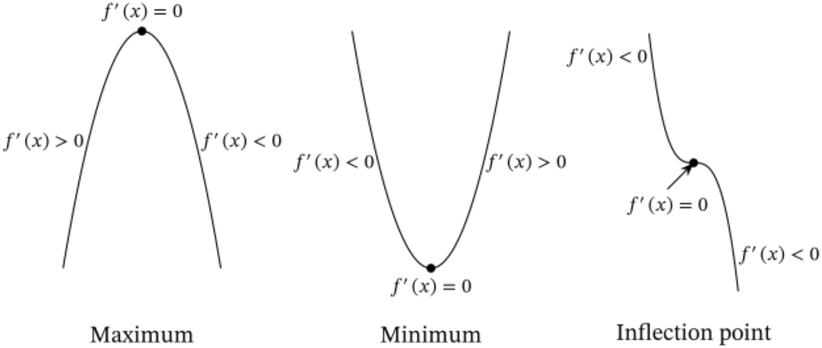


Sufficient Conditions for (Local) Extrema

First Derivative Test for a Local Extremum

Let f be a function defined on some interval containing the critical point $x = b$. Then, for sufficiently small $\varepsilon > 0$:

1. If $f'(b - \varepsilon) > 0$ and $f'(b + \varepsilon) < 0$ (i.e. $f'(x)$ switches signs from positive to negative as it crosses $x = b$), then at $x = b$ there is a **(local) maximum**.
2. If $f'(b - \varepsilon) < 0$ and $f'(b + \varepsilon) > 0$ (i.e. $f'(x)$ switches signs from negative to positive as it crosses $x = b$), then at $x = b$ there is a **(local) minimum**.
3. If $f'(b - \varepsilon) > 0$ and $f'(b + \varepsilon) > 0$ or $f'(b - \varepsilon) < 0$ and $f'(b + \varepsilon) < 0$ (i.e. $f'(x)$ does not change signs as it crosses $x = b$), then at $x = b$ there is a **point of inflection**.


$f'(x) = 0$
$f'(x) > 0$
$f'(x) < 0$
$f'(x) < 0$
$f'(x) > 0$
$f'(x) = 0$
$f'(x) < 0$
$f'(x) = 0$
$f'(x) < 0$
Maximum
Minimum
Inflection point

Sufficient Condition for (Local) Extrema

Second Derivative Test for a Local Extremum

Let f be a function defined on some interval containing the critical point $x = b$. In addition, $f'(b) = 0$ (i.e. b is a critical point) and f is twice differentiable at b.

- If $f''(b) < 0$, then f has a (local) **maximum** at b.
- If $f''(c) > 0$, then f has a (local) **minimum** at b.

$\longrightarrow$ Think about the relation this has with the sufficient conditions for a function to be convex/concave.

Exercise 5.11

Find the turning point of the function $x^a \ln(x)$ over the domain $x > 0$. Is it a maximum or a minimum? Does this depend on the value of a?

Exercise 5.7

Let

$$F(x) = \int_{e^{-x}}^{e^{x}} \frac{y}{1+y^2} dy$$

Find an expression for $F'(x)$.

Hint: Use the formula

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(y)dy\right] = f(v(x))v'(x) - f(u(x))u'(x)$$