5.3.

- f is differentiable
- b is a turning point (i.e. a maximum or a minimum)

Assume that b is a maximum, then,

$\Rightarrow f(b) \geq f(x)$ for $x \in [b-\delta, b+\delta]$

Recall Proposition 2.1. (subsection 2.3.1)

The following two statements are equivalent:

1. f is differentiable at b with derivative $f'(b)$.
2. As $h \to 0$, $f(b+h) = f(b) + h.f'(b) + o(h)$

With this in mind, let us show that it is not possible for $f'(b) > 0$ or $f'(b) < 0$ and hence conclude that $f'(b) = 0$.

- If $f'(b) > 0$:

(i) $h < 0$ $\quad f(b+h) = f(b) + \underbrace{h.f'(b)}_{<0} \Rightarrow f(b) > f(b+h)$

(ii) $h > 0$ $\quad f(b+h) = f(b) + \underbrace{h\,f'(b)}_{>0} \Rightarrow f(b) < f(b+h)$

Which contradicts our assumption of b being a maximum.

- If $f'(b) < 0$:

(i) $h < 0$ $\quad f(b+h) = f(b) + \underbrace{h f'(b)}_{>0} \Rightarrow f(b) < f(b+h)$

Which also contradicts our assumption of b being a maximum.

Therefore, we conclude that $f'(b) = 0$.

Recall that the term $o(h)$, by its definition, becomes negligible compared to $h f'(b)$ as $h \to 0$.

5.11.

$f(x) = x^a \cdot \ln(x) \qquad x > 0$

- $a \neq 0$

$f'(x) = a \cdot x^{a-1} \cdot \ln(x) + x^a \cdot \frac{1}{x} = x^{a-1} [1 + a\ln(x)]$

Set $f'(x) = 0$ and solve for x:

$\cancel{x^{a-1}} [1 + a\ln(x)] = 0 \ ; \ a\ln(x) = -1$

$\ln(x) = -\frac{1}{a} \ ; \ x = e^{-1/a}$

To classify this as a max/min we need to check which is the sign of $f''(e^{-1/a})$

$f''(x) = (a-1) \cdot x^{a-2} \cdot [1 + a\ln(x)] + x^{a-1} \cdot \frac{a}{x} =$

$= (a-1)\, x^{a-2} [1 + a\ln(x)] + a \cdot x^{a-2}$

then, for $x = e^{-1/a}$,

$f''(e^{-1/a}) = (a-1)\, e^{-\frac{1}{a}(a-2)} \left[1 + a \cdot \ln(e^{-1/a})\right] + a \cdot e^{-\frac{1}{a}(a-2)} =$

$= (a-1)\, e^{\frac{2}{a}-1} \underbrace{\left[1 + \underbrace{a \cdot \left(-\frac{1}{a}\right)}_{=-1}\right]}_{=0} + a \cdot e^{\frac{2}{a}-1} =$

$= a \cdot \underbrace{\exp\left(\frac{2}{a} - 1\right)}_{>0}$

Thus the turning point will be a max. if $a<0$ and a min. if $a>0$.

- $a=0$

$f(x) = x^0 \cdot \ln(x) = \ln(x)$

$f'(x) = \frac{1}{x} \longrightarrow$ no turning points over $x>0$ since for $x>0$, $f'(x)>0$.