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Exercise 1.8 If f and g are Riemann integrable, let $S_{\text{lower}}(f,n)$, $S_{\text{upper}}(f,n)$, $S_{\text{lower}}(g,n)$ and $S_{\text{upper}}(g,n)$ be the lower and upper Riemann sums for f and g respectively when calculating $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ using n sub-intervals.

(i) What could you use for the lower and upper Riemann sums for $\int_0^1 (f(x) - g(x))\, dx$

(ii) Can you use a limiting procedure as $n \to \infty$ to prove that

$$\int_0^1 (f(x) - g(x))\, dx = \int_0^1 f(x)dx - \int_0^1 g(x)dx \ ?$$

Solution:

- f and g are Riemann integrable over $(0,1)$ (note this doesn't mean that $f-g$ is Riemann integrable).
- $S_{\text{lower}}(f,n)$, $S_{\text{upper}}(f,n)$, $S_{\text{lower}}(g,n)$, $S_{\text{upper}}(g,n)$

Thus what we are being asked is similar to what the lecture notes examples provide but, in this case, for $a = 1$.

(i) Over the generic interval $\left(\frac{i-1}{n}, \frac{i}{n}\right)$ we know that:

$$\begin{cases} f_{i,\text{lower}} \le f(x) \le f_{i,\text{upper}} \\ g_{i,\text{lower}} \le g(x) \le g_{i,\text{upper}} \implies -g_{i,\text{upper}} \le -g(x) \le -g_{i,\text{lower}} \end{cases}$$

Therefore, over $\left(\frac{i-1}{n}, \frac{i}{n}\right)$, it is also true that:

$$f_{i,\text{lower}} - g_{i,\text{upper}} \le f(x) - g(x) \le f_{i,\text{upper}} - g_{i,\text{lower}}$$

Summing up $f_{i,\text{lower}} - g_{i,\text{upper}}$ over n we see that what we can use for the <u>lower Riemann sum</u> for $f - g$ is:

$$\frac{1}{n}\sum_{i=1}^{n}(f_{i,\text{lower}} - g_{i,\text{upper}}) \underset{\substack{\uparrow \\ \text{by linearity}}}{=} \frac{1}{n}\sum_{i=1}^{n} f_{i,\text{lower}} - \frac{1}{n}\sum_{i=1}^{n} g_{i,\text{upper}} = S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n)$$

Similarly, we see that what we can use for the <u>upper Riemann sum</u> for $f - g$ is:

$$\frac{1}{n}\sum_{i=1}^{n}(f_{i,\text{upper}} - g_{i,\text{lower}}) = S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n)$$

Let me briefly refer to the illustrative example that you have at the beginning of your lecture notes, that presented Riemann sums as the 'sum of rectangles': in here the width of each of our 'rectangles' is just the length of each of the n sub-intervals that we have over the interval $(0,1)$, which is $\frac{1-0}{n} = \frac{1}{n}$; the height of each 'rectangle' is just $f_{i,\text{upper}} - g_{i,\text{lower}}$, for $i = 1, ..., n$.

(ii) By construction of the Riemann sums we always have that:

$$S_{\text{lower}}(f,n) \leq \int_0^1 f(x)dx \leq S_{\text{upper}}(f,n) \tag{1}$$

Indeed, note that if you always choose the smallest value of the function on each interval, the Riemann sum $S_{\text{lower}}(f,n)$ must be an underestimate of the Riemann integral $\int_0^1 f(x)dx$. If you choose the largest value of the function on each interval, you will get an overestimate, $S_{\text{upper}}(f,n)$, of $\int_0^1 f(x)dx$.

Moreover, because f is Riemann integrable we know that:

$$\begin{cases} \lim_{n\to\infty} S_{\text{lower}}(f,n) = \int_0^1 f(x)dx \\ \lim_{n\to\infty} S_{\text{upper}}(f,n) = \int_0^1 f(x)dx \end{cases}$$

In other words, and using the 'alternative definition' provided in the slides, for any $\varepsilon > 0$ there exists some $n_0(f)$, such that for $n \geq n_0(f)$:

$$\begin{cases} \int_0^1 f(x)dx - S_{\text{lower}}(f,n) \leq \varepsilon/2 \implies \int_0^1 f(x)dx - \varepsilon/2 \leq S_{\text{lower}}(f,n) \\ S_{\text{upper}}(f,n) - \int_0^1 f(x)dx \leq \varepsilon/2 \implies S_{\text{upper}}(f,n) \leq \int_0^1 f(x)dx + \varepsilon/2 \end{cases} \tag{2}$$

Hence, combining 1 and 2, for $n \geq n_0(f)$ we have that:

$$\int_0^1 f(x)dx - \varepsilon/2 \leq S_{\text{lower}}(f,n) \leq \int_0^1 f(x)dx \leq S_{\text{upper}}(f,n) \leq \int_0^1 f(x)dx + \varepsilon/2 \tag{3}$$

Similarly, because g is Riemann integrable , for any $\varepsilon > 0$ we can find $n_0(g)$ such that for $n \geq n_0(g)$:

$$\int_0^1 g(x)dx - \varepsilon/2 \leq S_{\text{lower}}(g,n) \leq \int_0^1 g(x)dx \leq S_{\text{upper}}(g,n) \leq \int_0^1 g(x)dx + \varepsilon/2$$

or,

$$-\int_0^1 g(x)dx - \varepsilon/2 \leq -S_{\text{upper}}(g,n) \leq -\int_0^1 g(x)dx \leq -S_{\text{lower}}(g,n) \leq -\int_0^1 g(x)dx + \varepsilon/2 \tag{4}$$

Therefore, combining 3 and 4, we can conclude that as long as $n \geq \max\{n_0(f), n_0(g)\}$[1], we have that:

$$\begin{aligned} \int_0^1 f(x)dx - \int_0^1 g(x)dx - \varepsilon \leq \underbrace{S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n)}_{\text{Riemann lower sum of } f-g} \leq \\ \leq \int_0^1 f(x)dx - \int_0^1 g(x)dx \\ \leq \underbrace{S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n)}_{\text{Riemann upper sum of } f-g} \leq \int_0^1 f(x)dx - \int_0^1 g(x)dx + \varepsilon \end{aligned} \tag{5}$$

[1]Saying $n \geq \max\{n_0(f), n_0(g)\}$ is the same as saying 'for both $n \geq n_0(f)$ and $n \geq n_0(g)$'. And note 3 needs $n \geq n_0(f)$ to hold and 4 needs $n \geq n_0(g)$ to hold.

Note that we just got $\int_0^1 f(x)dx - \int_0^1 g(x)dx$ bounded by the Riemann lower and upper sums we derived in (i). Recall that the results obtained in (i) imply:

$$S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n) \leq \int_0^1 (f(x) - g(x)) \leq S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n) \tag{6}$$

Now, at this point we might be tempted to conclude that since:

$$S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n) \leq \int_0^1 (f(x) - g(x)) \leq S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n)$$

and

$$S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n) \leq \int_0^1 f(x)dx - \int_0^1 g(x)dx \leq S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n)$$

we can take the limit on both sides and by Riemann integrability conclude that since

$\lim_{n\to\infty} S_{\text{lower}}(f,n) - S_{\text{upper}}(g,n) = \lim_{n\to\infty} S_{\text{upper}}(f,n) - S_{\text{lower}}(g,n)$ it must be the case that

$$\implies \int_0^1 f(x)dx - \int_0^1 g(x)dx = \int_0^1 (f(x) - g(x))$$

However, note that the exercise tells us that f and g are Riemann integrable, but it does not tell us that $f - g$ is Riemann integrable! Therefore we cannot rely on this previous reasoning since this is assuming that $f - g$ is Riemann integrable. Indeed, $f - g$ is Riemann integrable, however when proving things we need to be very picky, and at each step rely only in things that we know for sure.

Of course we can try to prove that $f - g$ is Riemann integrable; nonetheless, and for the purposes of this exercise, it is much easier to simply combine 5 and 6 and note that we obtain:

$$\int_0^1 f(x)dx - \int_0^1 g(x)dx - \varepsilon \leq \int_0^1 (f(x) - g(x)) \leq \int_0^1 f(x)dx - \int_0^1 g(x)dx - \varepsilon$$

Hence, and since ε can be whatever positive number we want, letting $\varepsilon \to 0$, we get the desired result:

$$\int_0^1 f(x)dx - \int_0^1 g(x)dx = \int_0^1 (f(x) - g(x)) \,.$$