AS1056 - Mathematics for Actuarial Science. Chapter 3, Tutorial 2.

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Exercise 3.9

(i) Calculate the derivative of $f(x) = x^{-1}\ln(x) = \frac{\ln(x)}{x}$ over the domain $x > 0$.

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Answer:

$$f'(x) = -x^{-2}\ln(x) + x^{-1}x^{-1} = -x^{-2}\ln(x) + x^{-2} =$$
$$= \frac{1}{x^2}\left[1 - \ln(x)\right]$$

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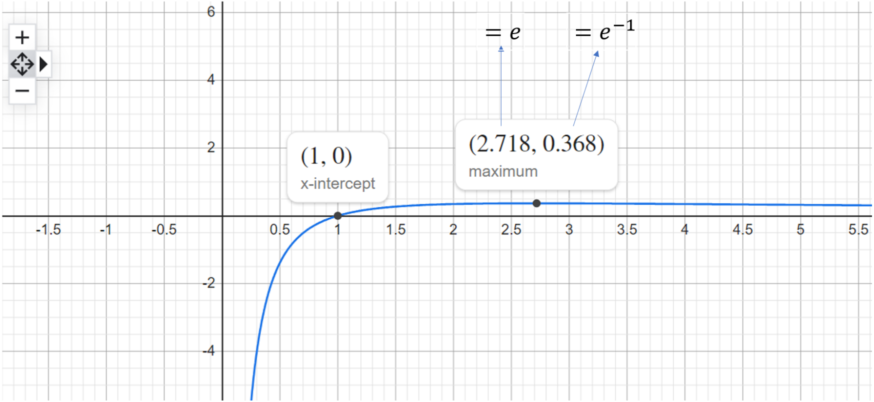
(ii) Sketch the graph of f.

Graph for $\ln(x)/x$



Let us check analytically that:

1. "As $x \to 0^+$, $f(x) \to -\infty$."
2. "f first reaches 0 at $x = 1$."
3. "f has a maximum at $x = e$."
4. "$f(x)$ is increasing for $x \in (0, e)$ and decreasing for $x \in (e, +\infty)$."
5. "$\lim_{x \to +\infty} f(x) = 0$."

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Summary

The function $f(x)$ is defined for all x in the interval $(0, +\infty)$. It increases from $-\infty$ to e^{-1} as x moves from 0 to e. $f(x)$ has a root at $x = 1$ and at $x = e$, $f(x)$ achieves its maximum value of e^{-1}. Then it decreases to 0 as x goes from e to $+\infty$.

Exercise 3.9

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Note that we can rewrite $x\ln(y) = y\ln(x)$ as $\frac{\ln(y)}{y} = \frac{\ln(x)}{x}$, i.e., as $f(y) = f(x)$, thus:

- It is clear that $y = x$ is always a solution
- Moreover, based on the properties of $f(x)$ that we have just discussed we'll be able to describe the behaviour of this new equation too.

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(iv) For which values of x does the equation $x\ln(y) = 2y\ln(x)$ have:

(a) no solutions

(b) one solution

(c) two solutions?

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(c) two solutions?

Let us rewrite $x\ln(y) = y\ln(x)$ as $\frac{\ln(y)}{y} = 2 \times \frac{\ln(x)}{x}$, i.e., $f(y) = 2 \times f(x)$ or $f(x) = \frac{1}{2}f(y)$.

Reconsider the intervals for x we've been analysing thus far:

- $x \in (0, 1]$
- $x \in (1, e)$ and $x \in (e, +\infty)$ and $x = e$

Exercise 3.5

Use the formula $\cos\left(\frac{\pi}{5}\right) = -\cos\left(2 \times \frac{2\pi}{5}\right)$ to obtain a cubic equation satisfied by the value of $\cos\left(\frac{\pi}{5}\right)$.

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Trigonometric identities

- $\cos(\pi - x) = -\cos(\pi) \implies \cos\left(\frac{\pi}{5}\right) = -\cos\left(2 \times \frac{2\pi}{5}\right)$ for $x = \frac{4\pi}{5}$
- $\cos(2x) = 2\cos^2(x) - 1$