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AS1056 - Chapter 3, Tutorial 2, 03-11-2023. Notes.

Hello everyone, I'd like to go over the exercise we discussed on Friday, and provide a detailed, step-by-step solution, especially for the more challenging sub-items (iii) and (iv). By meticulously working through the initial sub-items, I believe we can successfully tackle the entire exercise. So let's get to it.

Exercise 1.4

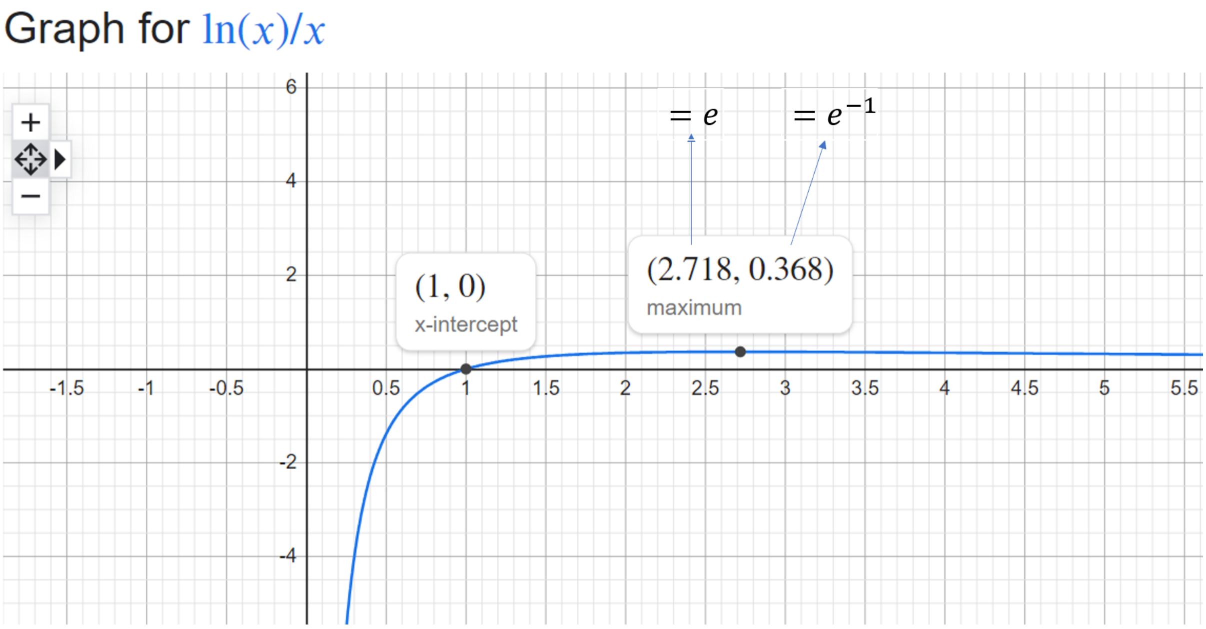
(i) Calculate the derivative of $f(x) = x^{-1}\ln(x) = \frac{\ln(x)}{x}$ over the domain $x > 0$.

Solution:

$$f'(x) = -x^{-2}\ln(x) + x^{-1}x^{-1} = -x^{-2}\ln(x) + x^{-2} = \frac{1}{x^2}\left[1 - \ln(x)\right]$$

(ii) Sketch the graph of f.

Solution: Use your favourite graphing calculator to plot $f(x) = \frac{\ln(x)}{x}$:



As you can observe the function $f(x)$ is defined for all x in the interval $(0, +\infty)$. It increases from $-\infty$ to e^{-1} as x moves from 0 to e. $f(x)$ has a root at $x = 1$ and achieves its maximum value of e^{-1} at $x = e$. Then it decreases very smoothly to 0 as x goes from e to $+\infty$.

Can we characterise the behaviour of $f(x)$ analytically without relying on its graph? Let's attempt to derive a description comparable to the visual interpretation we have just provided by proving the following properties analytically. Doing so will deepen our understanding of $f(x)$'s characteristics, and help us in addressing sub-items (iii) and (iv) of the exercise:

(a) **"As $x \to 0^+$, $f(x) \to -\infty$."** Recall that $\ln(x)$ is defined only for $x > 0$, hence we should look at the limit as $x \to 0^+$, i.e., as x approaches 0 from the right (indeed, because there are no values to the left of 0 in the domain of $\ln(x)$, the limit $x \to 0$ does not exist):

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty$$

Why $\frac{-\infty}{0^+} = -\infty$? Think about $\frac{-\infty}{0^+}$: on the one hand, a very big number over a very small number will be equal to a very big number; on the other hand, a negative number over a positive number will equal a negative number. For a more rigorous solution we would probably need to use the definition of the limit (ε's and δ's...).

Let us show also that $\lim_{x \to 0^+} \ln(x) = -\infty$. Note that,

$$\ln(x) = \ln\left(\frac{1}{1/x}\right) = \underbrace{\ln(1)}_{=0} - \ln\left(\frac{1}{x}\right) = -\ln\left(\frac{1}{x}\right). \text{ It is clear that } \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\Longrightarrow \lim_{x \to 0^+} \ln\left(\frac{1}{x}\right) = +\infty \text{ then, } \lim_{x \to 0^+} \ln(x) = \lim_{x \to 0^+} -\ln\left(\frac{1}{x}\right) =$$

$$\underset{\uparrow}{=} -\lim_{x \to 0^+} \ln\left(\frac{1}{x}\right) = -\infty$$

by linearity property of the limits

(b) **"f first reaches 0 at $x = 1$."** We just need to find the root(s) of x:

$$f(x) = \frac{\ln(x)}{x} = 0;\ \ln(x) = 0;\ e^{\ln(x)} = e^0 = 1;\ x = 1$$

(c) **"f has a maximum at $x = e$."**

$\longrightarrow$ First we find the critical points by setting $f'(x) = 0$ and solving for x:

$$\frac{1}{x^2}\left[1 - \ln(x)\right] = 0;\ 1 - \ln(x) = 0;\ \ln(x) = 1;\ e^{\ln(x)} = e^1;\ x = e^1 = e$$

At $x = e$, $f(x)$ takes the value:

$$f(x = e) = \frac{\ln(e)}{e} = \frac{1}{e} = e^{-1}$$

Thus, we have an critical point at $(x = e, y = e^{-1})$, and checking that $f''(x = e) < 0$ we conclude that $f(x)$ has a local maximum at (e, e^{-1}) (which indeed is the absolute maximum of $f(x)$).

(d) **"$f(x)$ is increasing for $x \in (0, e)$ and decreasing for $x \in (e, +\infty)$."**

- $f(x)$ is increasing for $x \in (0, e)$, i.e., $f'(x) > 0$ for $x \in (0, e)$. Note that for $x \in (0, e)$:

$$f'(x) = \underbrace{\frac{1}{x^2}}_{>0} \underbrace{\left[1 - \overbrace{\ln(x)}^{<1}\right]}_{>0} \longrightarrow \frac{1}{x^2} > 0 \text{ always, while } \ln(x) < 1 \text{ for } x \in (0, e).$$

Thus, $f'(x) > 0$ for $x \in (0, e)$.

- $f(x)$ is decreasing for $x \in (e, +\infty)$, i.e., $f'(x) < 0$ for $x \in (e, +\infty)$. Note that for $x \in (e, +\infty)$:

$$f'(x) = \underbrace{\frac{1}{x^2}}_{>0} \underbrace{\left[1 - \overbrace{\ln(x)}^{>1}\right]}_{<0} \longrightarrow \frac{1}{x^2} > 0 \text{ always, while } \ln(x) > 1 \text{ for } x \in (e, +\infty).$$

  Thus, $f'(x) < 0$ for $x \in (e, +\infty)$.

Note: If you don't see why $\ln(x) < 1$ for $x \in (0, e)$ and $\ln(x) > 1$ for $x \in (e, +\infty)$ just recall that $\ln(e) = 1$ and that $\ln(x)$ is an increasing function[1].

(e) **"$\lim_{x \to +\infty} f(x) = 0$."**

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\ln(x)}{x} \underset{\substack{\uparrow \\ \text{l'Hôpital}}}{=} \lim_{x \to +\infty} \frac{1/x}{1} = 0$$

(iii) For which values of x is there more than one value of y which satisfies the equation $x \ln(y) = y \ln(x)$?

Solution: Note that we can rewrite $x \ln(y) = y \ln(x)$ as $\frac{\ln(y)}{y} = \frac{\ln(x)}{x}$, i.e., as $f(y) = f(x)$, thus:

- The equation $f(y) = f(x)$ is symmetric in x and y: any point (x, y) that lies on the curve will also have its reflection point (y, x) lie on the curve.
- When $x = y$, the equation $f(y) = f(x)$ is trivially satisfied.
- Moreover, based on the properties of $f(x)$ that we have just discussed we'll be able to describe the behaviour of this new equation too.

Hence, given that we have already thoroughly worked out the properties of $f(x)$ on the previous sub-items, I will suggest you to answer this sub-item analytically (using what we know about $f(x)$), and afterwards check the conclusions obtained sketching a graph of the equation.

Recall that $f(x)$ is a function that starts from $-\infty$ when x is close to 0, crosses the x-axis at $x = 1$, achieves its maximum value at $x = e$ and then smoothly decays to 0 as x grows larger and larger. Let us then consider the behaviour of $f(x)$ —and then infer the behaviour of $f(y) = f(x)$—, on the following intervals of x: $(0, 1]$; $(1, e)$ and $(e, +\infty)$; $\{e\}$.

1. For $x \in (0, 1]$ there's no other solution rather than $y = x$, since we know that:
   - $f(x)$ has a root at $x = 1$
   - $f(x) \leq 0$ for $x \in (0, 1]$ and $f(x) > 0$ for $x \in (1, +\infty)$
   - $f'(x) \geq 0$ for $x \in (0, 1]$ (i.e., monotonicity on (0,1])

   For x within $(0, 1]$, the function $f(x) = \frac{\ln(x)}{x}$ is negative and increasing, where $f(x)$ crosses from negative to positive at $x = 1$. As a result, there is no $x_1 > 1$ such that $f(x_1) = f(x_0)$ for x_0 in $(0, 1]$ since $f(x)$ is negative for $x < 1$ and

[1] $\frac{d \ln(x)}{dx} = \frac{1}{x} > 0$ since $\ln(x)$ is only defined for $x > 0$.

positive for $x > 1$. In addition, the monotonic nature of $f(x)$ in $(0, 1]$ tells us that $f(x)$ is a one-to-one function in this interval. In other words, since $f(x)$ is increasing for $x \in (0, 1]$ we know that there's no $x_0' \neq x_0$, $x_0' \in (0, 1]$, such that $f(x_0') = f(x_0)$. Therefore, $f(y) = f(x)$ is also a one to one mapping, i.e., the only pair that satisfies the equation $f(y) = f(x)$ for each x in $(0, 1]$ is when y is exactly x. This means the equation $x \ln(y) = y \ln(x)$ has a unique solution $y = x$ in this interval.

2. $x \in (1, e)$ and $x \in (e, +\infty)$.
   - $x \in (1, e) \implies$
     $f(x) \in \left(f(1) = 0, f(e) = e^{-1}\right);\ f'\left((1, e)\right) > 0$ (increasing)
   - $x \in (e, +\infty) \implies$
     $f(x) \in \left(f(e) = e^{-1}, \lim_{x \to +\infty} f(x) = 0\right);\ f'\left((e, +\infty)\right) < 0$ (decreasing)

   Given the above and that we know there's a maximum at $x = e$ it is clear that for every $x_0 \in (1, e)$ there exists one $x_1 \in (e, +\infty)$ such that $f(x_0) = f(x_1)$. Thus:

   ➢ For values $x_0 \in (1, e)$ there are two values of y that satisfy the equation $x \ln(y) = y \ln(x)$:
     - $y_0 \in (1, e)$, in fact $y_0 = x_0$
     - $y_1 \in (e, +\infty)$

   ➢ And vice versa, for values $x_1 \in (e, +\infty)$ there are two values of y that satisfy the equation $x \ln(y) = y \ln(x)$:
     - $y_0 \in (1, e)$
     - $y_1 \in (e, +\infty)$, in fact $y_1 = x_1$

3. For $x \in \{e\}$, i.e., $x = e$, there's no other solution rather than $y = x$.

   $\longrightarrow f(x = e) = \dfrac{\ln(e)}{e} = \dfrac{1}{e} = e^{-1}$ and the only y value such that,
   $f(y) = f(x = e) = e^{-1}$ is clearly $y = e$.

(iv) For which values of x does the equation $x \ln(y) = 2y \ln(x)$ have:

(a) no solutions

(b) one solution

(c) two solutions?

Solution: Let us rewrite $x \ln(y) = y \ln(x)$ as $\frac{\ln(y)}{y} = 2 \times \frac{\ln(x)}{x}$, i.e., $f(y) = 2 \times f(x)$ or $f(x) = \frac{1}{2} f(y)$.

Reconsider the intervals for x we've been analysing thus far:

1. $x \in (0, 1]$
2. $x \in (1, e)$ and $x \in (e, +\infty)$ and $x = e$

1. As we have checked in the previous sub-items for $x \in (0, 1]$ there is one and only one solution for $f(x)$ in this case $\implies$ given that $f(y) = 2f(x)$ then there is also one and only one solution for $f(y)$. In other words, for $x \in (0, 1]$ there is one

and only one y (and you can check that $y \in (0,1]$ for $x \in (0,1]$)[2] which satisfies $x\ln(y) = 2y\ln(x)$.

2. $x \in (1,e)$ and $x \in (e,+\infty)$ and $x = e$

   Note that taken by separate both $f(x)$ and $f(y)$ range from $-\infty$ up to e^{-1}. However, going back to $f(y) = 2f(x)$ we notice that:

   - If $f(x) = -\infty \implies f(y) = 2\times(-\infty) = -\infty$, which is fine.
   - However, if $f(x) = e^{-1} \implies f(y) = 2e^{-1}$, but this cannot happen since $f(y) \in (-\infty, e^{-1}]$

   So, implicitly, $f(y) = 2f(x)$, i.e., $f(x) = \frac{1}{2}f(y)$ is telling us that for $x\ln(y) = 2y\ln(x)$ to hold we need that $f(x)$ ranges from $-\infty$ up to $\frac{1}{2}e^{-1}$. And of course, given the characteristics of $f(x)$ that we have already studied, we know that there will be two values of x such that $f(x) = \frac{1}{2}e^{-1}$. In particular, there is one value $x_0 \in (1,e)$ such that $f(x_0) = \frac{1}{2}e^{-1}$, and another value $x_1 \in (e,+\infty)$ such that $f(x_1) = \frac{1}{2}e^{-1}$.[3]

   Hence, $f(y) = 2\times f(x) \implies f(x) \in \left(-\infty, \frac{1}{2}e^{-1}\right]$. So, let us redefine the intervals of x on which to analyse the behaviour of $f(y) = 2\times f(x)$:

   (1) $1 < x < x_0 = 1.261070487$
   (2) $x_0 = 1.261070487 < x < x_1 = 14.56100391$
   (3) $x > x_1 = 14.56100391$
   (4) $x = x_0 = 1.261070487$; $x = x_1 = 14.56100391$

   (2) If $x \in (x_0, x_1)$ there is no solution to the equation since $f(y)$ cannot exceed e^{-1}.

   (4) $x = x_0 \implies f(x_0) = \frac{1}{2}e^{-1}$ and $x = x_1 \implies f(x_1) = \frac{1}{2}e^{-1}$. In this case there is only one value of y that makes $f(x) = \frac{1}{2}f(y)$ hold. This value is $y = e$, which makes $f(y) = e^{-1}$.

   (1) and (3): for $1 < x < x_0$ and $x > x_1$ there are two values of y that satisfy the equation $f(y) = 2f(x)$. This because of the characteristics of $f(x)$ and that in these range of values of x no violation of the range of $f(y)$ occurs.

Remember to plot the equations of sub-items (iii) and (iv) and check the results!

[2] $\lim_{x\to 0^+} f(x) = -\infty \implies f(y) = 2\times(-\infty) \implies y \to 0^+$ and $f(x=1) = 0 \implies f(y) = 2\times 0 \implies y = 1$.

[3] The equation $\frac{1}{2}e^{-1} = \frac{\ln(x)}{x}$ has not closed form solution, but you can approximate the values of x_0 and x_1 using some approximation method such as Newton-Raphson (we'll see this on the 2nd term). In particular $x_0 = 1.261070487$ and $x_1 = 14.56100391$.