2.8

(ii) $f_2(x) = \sqrt{1+x^2}$ as $x \to 0$

(1) Compute $f_2(0)$:

$$f_2(0) = \sqrt{1+0^2} = 1$$

(2) Compute $f_2'(0)$:

$$f_2'(x) = \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x$$

$$f_2'(0) = \frac{1}{2}(1+0^2)^{-1/2} \cdot 2 \cdot 0 = 0$$

Proposition 2.1. tells us that, since f is differentiable at $x_0 = 0$ with derivative $f_2'(0)$, then, as $h \to 0$, $f_2(0+h) = f_2(0) + h f_2'(0) + o(h)$.

Hence,

$$f_2(h) = 1 + h \cdot 0 + o(h) = \underline{1 + o(h)}$$

$f_2(0) = 1$ and $f_2'(0) = 0$ (i.e. the function is flat at $x = 0$ ⇝ the function doesn't change much as x approaches 0 and indeed what we have just shown is that this change is strictly slower than the change in x around $x = 0$.

• Using the definition

$f_2(x) = 1 + o(x)$ as $x \to 0$ which is the same

as saying $f_2(x) - 1 = o(x)$ as $x \to 0$.

By definition,

$$f_2(x) - 1 = o(x) \iff \lim_{x \to 0} \frac{f_2(x) - 1}{x} = 0$$
as $x \to 0$

Thus, want to show that $\lim_{x \to 0} \frac{f_2(x) - 1}{x} = 0$

$$\lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{x} \underset{\substack{\downarrow \\ \text{l'Hôpital}}}{=} \lim_{x \to 0} \frac{(1/2)(1+x^2)^{-1/2} \cdot 2x}{1} = 0$$

• Big O notation

If we followed Proposition 2.1 we would get that we can express $f_2(x)$ as $f_2(x) = 1 + O(x)$

$f_2(x) = 1 + O(x^2)$ as $x \to 0$ i.e.

$f_2(x) - 1 = O(x^2)$ as $x \to 0$

By definition,

$f_2(x) - 1 = O(x^2)$ as $x \to 0$ $\iff$ $\left| \frac{f_2(x) - 1}{x^2} \right| \leq M$ for sufficiently small x.

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x^2} \underset{\text{l'Hôpital}}{=} \lim_{x \to 0} \frac{(1/2)(1+x^2)^{-1/2} \cdot 2x}{2x} = \frac{1}{2}$$

(iii) $f_3(x) = \frac{x}{x^2+1}$ as $x \to 3$

(1) Compute $f_3(3)$:

$$f_3(3) = \frac{3}{9+1} = \frac{3}{10} = 0.3$$

(2) Compute $f_3'(3)$:

$$f_3'(x) = \frac{(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$f_3'(3) = \frac{1-9}{(9+1)^2} = -\frac{8}{100} = -0.08$$

then, by proposition 2.1.

$$f_3(3+h) = f_3(3) + h \cdot f_3'(3) + o(h) \quad \text{as } h \to 0$$

since f_3 is differentiable at $x_0 = 3$ with derivative $f_3'(3)$.

$$f_3(3+h) = 0.3 - 0.08 \cdot h + o(h)$$

or alternatively, using big-O notation,

$$f_3(3+h) = 0.3 + O(h)$$

(iv) $f_4(x) = \dfrac{x-8}{(x+2)(2x-1)}$ as $x \to \frac{1}{2}$

(1) Compute $f_4(\frac{1}{2})$:

$$f_4\left(\tfrac{1}{2}\right) = \frac{0.5-8}{(0.5+2)(2\cdot 0.5-1)} = \frac{-7.5}{2.5 \times 0} = \frac{-7.5}{0} = -\infty$$

...

Remember we're interested on the behaviour of $f_4(x)$ as x approaches $\frac{1}{2}$. Thus consider:

$$f_4\left(\tfrac{1}{2}+h\right) = \frac{\frac{1}{2}+h-8}{(\frac{1}{2}+h+2)(1+2h-1)} = \underbrace{\frac{1}{2h}}_{\substack{\to\infty \\ \text{as } h\to 0}} \cdot \underbrace{\frac{h-7.5}{h+2.5}}_{\substack{<\infty \\ \text{as } h\to 0}}$$

The growth of the function is dominated by the $\frac{1}{2h}$ term as h gets close to 0 (i.e. as x approaches $\frac{1}{2}$). Thus, $f_4(\frac{1}{2}+h) = O(h^{-1})$ as $h \to 0$.