AS1056 - Mathematics for Actuarial Science. Chapter 18, Tutorial 2.

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The total differential

$$df = f_x dx + f_y dy + f_z dz \tag{1}$$

- Foundational concept in multivariate differential calculus.
- It expresses an infinitesimally small change in f as a linear combination of infinitesimally small changes in the variables x, y, and z, i.e., dx, dy, dz.

Approximation using differentials

Note that while the total differential provides an exact measure for infinitesimally small changes, we often need to approximate changes over finite intervals...

The value that a function of three variables $f(x, y, z)$ takes at some point $(x, y, z) = (x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ can be approximated by:

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) \approx & f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)\Delta x \\ & + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z \end{aligned} \tag{2}$$

- This is straightforward to see by just considering the **first-order (i.e. linear) Taylor approximation in 3 dimensions** (check Section 18.6.1 of Lecture Notes).

- Alternatively —following the approach in Section 18.6 of the lecture notes—, expression 2 can also be derived by taking,

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0), \text{ i.e.,}$$
$$f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) = f(x_0, y_0, z_0) + \Delta f \tag{3}$$

and then approximating Δf —using the **definition of partial derivative**— via $\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z$. Replacing in equation 3:

$$\longrightarrow f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) = f_x(x_0, y_0, z_0) + \Delta f \approx$$
$$\approx f_x(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y +$$
$$+ f_z(x_0, y_0, z_0)\Delta z$$

How $\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z$?

Note that the definition of partial derivative tells us that:

$$f_x(x_0, y_0, z_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

then,

$$\longrightarrow f_x(x_0, y_0, z_0) \approx \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}, \quad \text{for } \Delta x \text{ small.}$$

Therefore we have that:

- $f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0) \approx \Delta x \times f_x(x_0, y_0, z_0)$
- $f(x_0, y_0 + \Delta y, z_0) - f(x_0, y_0, z_0) \approx \Delta y \times f_y(x_0, y_0, z_0)$
- $f(x_0, y_0, z_0 + \Delta z) - f(x_0, y_0, z_0) \approx \Delta z \times f_z(x_0, y_0, z_0)$

It is interesting to also notice that the definition of first partial derivative is closely related to the concept first-order (linear) Taylor approximation. Take a first-order Taylor approximation of $f(x+\Delta x, y, z)$ at (x_0, y_0, z_0):

$$f(x,y,z) \approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)\Delta x + f_y(x_0,y_0,z_0)\Delta y + \\ + f_z(x_0,y_0,z_0)\Delta z$$

now holding y and z constant (i.e., $y = y_0\ \forall y$, $z = z_0\ \forall z$), and given that $x = x_0 + \Delta x$ (since $\Delta x = x - x_0$):

$$f(x_0+\Delta x, y_0, z_0) \approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)\Delta x.$$

Re-arranging this expression we obtain:

$$f_x(x_0,y_0,z_0) \approx \frac{f(x,y,z) - f(x_0,y_0,z_0)}{\Delta x}$$

and letting $\Delta x \to 0$, we get the equality (note that the terms involving $(\Delta x)^n,\ n > 1$ in the Taylor expansion will go faster to 0 than the linear term $f_x(x_0,y_0,z_0) \times \Delta x$):

$$f_x(x_0,y_0,z_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0, z_0) - f(x_0,y_0,z_0)}{\Delta x}$$

Let us get back to the approximation of Δf. Operating and replacing in Δf we have that:

$$
\begin{aligned}
&\Delta f = f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0, z_0) = \\
&= \underbrace{f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0+\Delta y, z_0+\Delta z)}_{\approx \Delta x \times f_x(x_0, y_0+\Delta y, z_0+\Delta z)} + \\
&+ \underbrace{f(x_0, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0, z_0+\Delta z)}_{\approx \Delta y \times f_y(x_0, y_0, z_0+\Delta z)} + \\
&+ \underbrace{f(x_0, y_0, z_0+\Delta z) - f(x_0, y_0, z_0)}_{\approx \Delta z \times f_z(x_0, y_0, z_0)} \approx \\
&\approx \Delta x \times f_x(x_0, y_0+\Delta y, z_0+\Delta z) + \Delta y \times f_y(x_0, y_0, z_0+\Delta z) + \\
&+ \Delta z \times f_z(x_0, y_0, z_0) \quad \text{for } \Delta x, \Delta y, \Delta z \text{ small.}
\end{aligned}
$$

Assuming that the first partial derivatives are continuous, we have that:

- $f_x(x_0, y_0 + \Delta y, z_0 + \Delta z) \to f_x(x_0, y_0, z_0)$ as $\Delta y \to 0$, $\Delta z \to 0$
- $f_y(x_0, y_0, z_0 + \Delta z) \to f_y(x_0, y_0, z_0)$ as $\Delta z \to 0$

Then, letting $\Delta y \to 0$, $\Delta z \to 0$, we finally get:

$$\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z$$

The above expression approximates the change in the function f for finite changes in the variables (Δx, Δy, Δz).
In the limit, $\Delta x, \Delta y, \Delta z \to 0$, (by convention) this becomes the **total differential**:

$$df = f_x dx + f_y dy + f_z dz$$

The total differential is a foundational concept of multivariate differential calculus, in particular, it provides the exact change in f for infinitesimally small changes (dx, dy, dz).

For further intuition, let me rewrite the total differential formula as:

$$\frac{df}{dx} = fx\frac{dx}{dx} + fy\frac{dy}{dx} + fz\frac{dz}{dx} = fx + fy\frac{dy}{dx} + fz\frac{dz}{dx} \quad \text{(Chain Rule)}$$

The effect that an infinitesimal change of x has in f is equal to the *direct* effect that this change of x has into f, plus the *indirect* effect that this change of x has into f through y and z.

Example:

f: The premium amount for the life insurance policy.
x: Mortality risk
y: Expenses (administrative and operational)
z: Investment return

- *Direct Effect:* An increase in mortality risk means the insurer is more likely to make a payout, which directly increases the premium required to cover this risk.
- *Indirect Effect through Expenses:* Higher mortality risk can lead to increased claim processing costs, indirectly raising the premium needed to cover these additional operational expenses.
- *Indirect Effect through Investment Return:* A rise in mortality risk could necessitate a more conservative investment strategy to ensure funds are available for potential claims, possibly reducing investment income and indirectly affecting the premium calculation.

Exercise 18.10

Use the definition of differential to work out the approximate value of the number

$$101^3\sqrt{98}\cos(\pi + 0.1).$$

You may do this by using the definition of the differential of a function of three variables $f(x, y, z) = x^3\sqrt{y}\cos(z)$:

$$df = fxdx + fydy + fzdz.$$

Maximums, minimums and saddle points

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1. *Stationary Points*. To locate the stationary points of a multivariate function, we take first partial derivatives and equate to zero. In other words, set $\nabla f = \mathbf{0}$.
2. *Classify Stationary Points*. Calculate the eigenvalues of the Hessian at each stationary point:

$$\det(\mathcal{H}(f) - \lambda I) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} - \lambda & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} - \lambda \end{vmatrix} = (f_{xx} - \lambda)(f_{yy} - \lambda) - f_{xy}^2 = 0$$

- If $\lambda_1 > 0$ and $\lambda_2 > 0$: (local) minimum.
- If $\lambda_1 < 0$ and $\lambda_2 < 0$: (local) maximum.
- If $\operatorname{sign} \lambda_1 \neq \operatorname{sign} \lambda_2$: saddle point.

Exercise 18.12

Find all the stationary points of the function

$$f(x, y) = (x + y)^4 - x^2 - y^2 - 6xy$$

and identify their type.