Emilio Luis Sáenz Guillén
PhD in Actuarial Science Student

AS1056 - Chapter 18, Tutorial 2. 11-04-2024. Notes.

Exercise 18.10: *Approximation by differentials.*

- Approximate the value $101^3\sqrt{98}\cos(\pi+0.1)$ using the definition of differential.

First, let us define:

$$f(x,y,z) := x^3\sqrt{y}\cos(z) \tag{1}$$

then, by first-order (i.e. linear) Taylor approximation in three dimensions:

$$f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) \approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)\Delta x + f_y(x_0,y_0,z_0)\Delta y + f_z(x_0,y_0,z_0)\Delta z$$

For our purposes, let us have:

$$x_0 = 100; \Delta x = 1; y_0 = 100; \Delta y = -2; z_0 = \pi; \Delta z = 0.1$$

then,

$$\begin{aligned}\longrightarrow f(101,98,\pi+0.1) \approx & f(100,100,\pi) + f_x(100,100,\pi)\times 1 + f_y(100,100,\pi)\times(-2) + \\ & + f_z(100,100,\pi)\times 0.1\end{aligned}$$

Thus, let us calculate the values we need:

- $f(100,100,\pi) = 100^3\sqrt{100}\cos(\pi) = -10,000,000$
- $f_x = 3x^2\sqrt{y}\cos(z)$; $f_y = \frac{x^3}{2\sqrt{y}}\cos(z)$; $f_z = -x^3\sqrt{y}\sin(z)$
- $f_x(100,100,\pi) = 3\times 100^2\sqrt{100}\cos(\pi) = -300,000$; $f_y(100,100,\pi) = \frac{100^3}{2\sqrt{100}}\cos(\pi) = -50,000$; $f_z(100,100,\pi) = -100^3\sqrt{100}\underbrace{\sin(\pi)}_{=0} = 0$

Therefore,

$$f(101,98,\pi+0.1) \approx -10,000,000 - 300,000 + 100,000 + 0 = -10,200,000$$

which is relatively close to the exact value $10,200,000$.

Exercise 18.12

$$f(x,y) = (x+y)^4 - x^2 - y^2 - 6xy$$

1. *Stationary Points.* To find the stationary points of f, we take the first partial derivatives and equate to 0:

$$\begin{cases} f_x = 4(x+y)^3 - 2x - 6y = 0 \\ f_y = 4(x+y)^3 - 2y - 6x = 0 \end{cases}$$

Now, take $f_x = f_y$:

$$\cancel{4(x+y)^3} - 2x - 6y = \cancel{4(x+y)^3} - 2y - 6x; \quad 4x = 4y; \quad x = y$$

Replacing $x = y$ in $f_x = 0$:

$4(x+x)^3 - 2x - 6x = 32x^3 - 8x = 0$

$4x^3 - x = 0; \quad x(4x^2 - 1) = 0$

Then,

$\longrightarrow x_1 = 0$

$\longrightarrow 4x^2 = 1; \quad x^2 = \dfrac{1}{4}; \quad x_{2,3} = \pm\sqrt{\dfrac{1}{4}} = \pm\dfrac{1}{2}$

Given that $x = y$, the three stationary points of f are: $(0,0)$; $\left(\frac{1}{2}, \frac{1}{2}\right)$; $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.

2. *Classify Stationary Points.* To classify these stationary points we need to compute the Hessian matrix of second partial derivatives and solve the corresponding characteristic equation for each of the stationary points:

$$\det(\mathcal{H}(f) - \lambda I) = \begin{vmatrix} f_{xx} - \lambda & f_{xy} \\ f_{yx} & f_{yy} - \lambda \end{vmatrix} = (f_{xx} - \lambda)(f_{yy} - \lambda) - f_{xy}^2 = 0$$

Let us calculate the second derivatives:

$$f_{xx} = 12(x+y)^2 - 2; \quad f_{xy} \underset{\substack{\uparrow \\ \text{by Theorem 18.1}}}{=} f_{yx} = 12(x+y)^2 - 6; \quad f_{yy} = 12(x+y)^2 - 2$$

- $(0,0)$: $f_{xx}(0,0) = -2 = f_{yy}$; $f_{xy}(0,0) = f_{yx}(0,0) = -6$
  Thus,

$$\begin{vmatrix} -2 - \lambda & -6 \\ -6 & -2 - \lambda \end{vmatrix} = (-2 - \lambda)^2 - 36 = 0$$

$$\lambda^2 + 4\lambda + 4 - 36 = \lambda^2 + 4\lambda - 32 = 0$$
$$\longrightarrow \lambda_1 = 4$$
$$\longrightarrow \lambda_2 = -8$$

  That is, $\operatorname{sign}\lambda_1 \neq \operatorname{sign}\lambda_2$ and therefore $(0,0)$ is a saddle point.

- $\left(\frac{1}{2}, \frac{1}{2}\right)$: $f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) = 10 = f_{yy}$; $f_{xy}\left(\frac{1}{2}, \frac{1}{2}\right) = f_{yx}\left(\frac{1}{2}, \frac{1}{2}\right) = 6$ Then, the characteristic equation of the Hessian matrix of f at $\left(\frac{1}{2}, \frac{1}{2}\right)$:

  $$\begin{aligned}
  &(10-\lambda)^2 - 36 = 100 - 20\lambda + \lambda^2 - 36 = \lambda^2 - 20\lambda + 64 = 0 \\
  &\longrightarrow \lambda_1 = 16 \\
  &\longrightarrow \lambda_2 = 4
  \end{aligned}$$

  Therefore, $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a (local) minimum. And same goes for $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.