AS1056 - Mathematics for Actuarial Science. Chapter 17, Tutorial 2.

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Numerical methods for finding roots

In this tutorial we discuss the problem of finding approximate solutions of the equation

$$f(x) = 0 \tag{1}$$

- In some cases it is possible to find the *exact roots* of the equation 1, for example, when $f(x)$ is a quadratic or cubic polynomial.
- Otherwise, in general, one is interested in finding *approximate solutions* using some (numerical) methods
    1. **Fixed-Point/Function Iteration Method)**
    2. **Newton-Raphson**

Example:

Consider $f(x) = e^x - 3x$, how do you solve for x?

$\longrightarrow$ This equation does not have a closed form solution. This means we can't solve it just through algebraic manipulation. Instead, we must employ some numerical method to find an approximate solution.

Fixed-Point/Function Iteration Method

In this method, we first rewrite equation 1 as:

$$x = g(x) \tag{2}$$

Therefore, any solution of equation 2, which is a fixed-point of g —i.e. a point s.t. $x = g(x)$ holds—, is a solution of equation 1. Since, if x^* is a solution of $x = g(x)$, then, $f(x^*) = x^* - g(x^*) = 0$.

Then consider the following algorithm...

Algorithm 1 Fixed-Point/Function Iteration

Start from any point x_0 and consider the recursive process

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \dots \tag{3}$$

which gives rise to the sequence x_0, x_1, x_2, $\dots$ of iterated function applications x_0, $g(x_0)$, $g(g(x_0))$, $\dots$

If f is continuous and the sequence (x_n) converges to some x^*, then it is clear that x^* is a fixed-point of g and hence it is a solution of equation 1.

Theorem

Let $g : [a, b] \to [a, b]$ be a differentiable function such that

$$|g'(x)| \leq k < 1 \text{ for all } x \in [a, b]. \tag{4}$$

Then g has exactly one fixed-point x^* in $[a, b]$ and the sequence (x_n) defined by the process 3, with starting point $x_0 \in [a, b]$, converges to x^*.

Newton-Raphson

If f is differentiable, $f'(x_0) \neq 0$, and the initial guess is close to the solution, then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is a better approximation of the root than x_0. Then, iterating...

Algorithm 2 Newton-Raphson

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots$$

until a sufficiently precise value is reached.

Note that we can write $g(x) = x - \frac{f(x)}{f'(x)}$ then Algorithm 2 is a particular case of Algorithm 1. That is, convergence is guaranteed if $|g'(x)| < 1$. In particular,

$$|g'(x)| = \left|1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2}\right| = \left|\frac{\cancel{f'(x)^2} - \cancel{f'(x)^2} + f(x)f''(x)}{f'(x)^2}\right|$$

$$= \left|\frac{f(x)f''(x)}{f'(x)^2}\right| < 1 \implies \boxed{|f(x)f''(x)| < |f'(x)|^2}$$

Please, keep in mind that Newton-Raphson requires $f'(x) \neq 0$.

Fixed-Point Iteration Method vs. Newton-Raphson

Sufficient but not necessary

The *convergence conditions* we have presented for the Fixed-Point/Function Iteration Method and the Newton-Raphson Method respectively are sufficient but not necessary. In other words, convergence is guaranteed if we pick an initial point x_0 that fulfils the respective convergence conditions. However, this doesn't mean that convergence cannot occur for some other x_0 that does not fulfil these conditions.

Fixed-Point Iteration Method vs. Newton-Raphson

- Fixed-Point Method: If $g'(x^*) \neq 0$, the sequence converges *linearly* to the fixed-point x^*; if $g'(x^*) = 0$, the convergence is at least *quadratic*.
- Newton-Raphson's method *converges quadratically*.

Exercise 17.10

(i) Sketch the function $f(x) = e^x - 3x$ on the domain $0 \leq x \leq 5$. How many roots does it have?

(ii) Find the function roots using the fixed-point/function iteration method. Draw a function iteration diagram.

(iii) Find the function roots using Newton-Raphson method.