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AS1056 - Chapter 17, Tutorial 2. 04-04-2024. Notes.

Let me note that there's a mistake on 17.10 (ii) (a) of the additional exercises solutions (and actually also on the stament of the exercise). Instead follow the solutions below.

Exercise 17.10

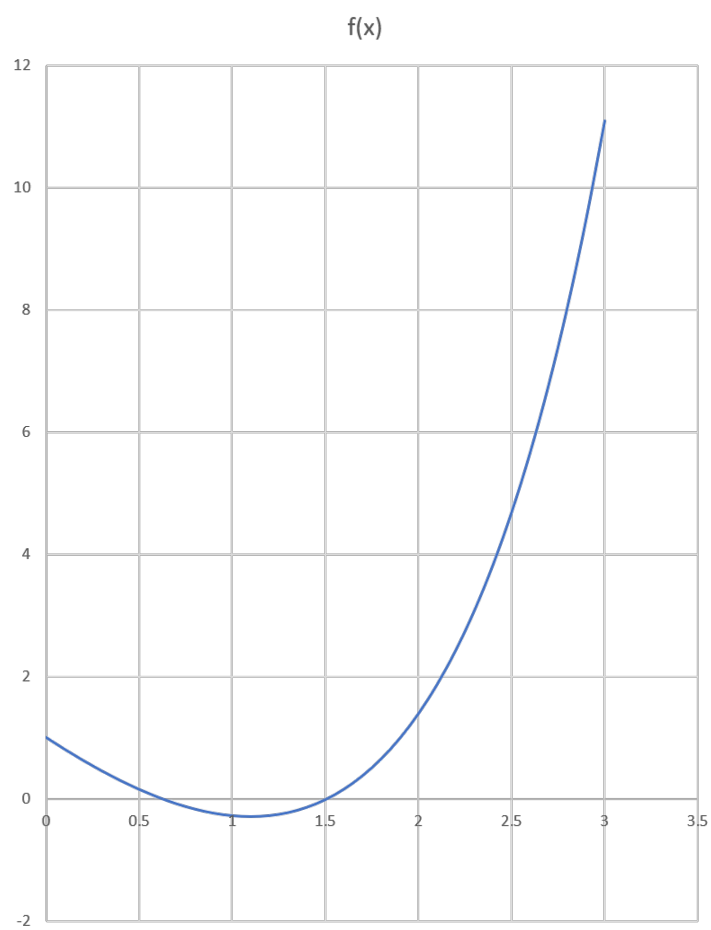
(i) Sketch the function $f(x) = e^x - 3x$ on the domain $0 \leq x \leq 5$. How many roots does it have?

(ii) Find the function roots using the fixed-point/function iteration method. Draw a function iteration diagram.

(iii) Find the function roots using Newton-Raphson method.

$$f(x) = e^x - 3x; \quad 0 \leq x \leq 5$$

(i) The function $f(x)$ has two roots.



(ii) **Fixed-Point/Function iteration method.**

First, set $f(x) = 0$, then:

$e^x - 3x = 0; \quad e^x = 3x$

now, note there is no unique form to re-express $f(x) = 0$ as $x = g(x)$. For instance, consider:

$$\longrightarrow x = \frac{1}{3}e^x = g_1(x); \quad g_1'(x) = \frac{1}{3}e^x$$
$$\longrightarrow x = \ln(3x) = g_2(x); \quad g_2'(x) = \frac{1}{x}$$

Now given the sketch of $f(x)$ we depicted in (i), let us pick some value close to each of the roots and test the convergence condition $|g'(x)| < 1$. This will allow us to decide which function g_1 or g_2 is preferable in order to converge to the first and second root respectively.

- So, the first root seems sufficiently close to $x_0 = 0.5$

  $$\longrightarrow g_1'(0.5) = \frac{1}{3}e^{0.5} = 0.5495737569 < 1 \quad ✓$$
  $$\longrightarrow g_2'(0.5) = \frac{1}{0.5} = 2 > 1 \quad ✗$$

  Therefore, to find the first root we iterate using $g_1(x)$:

  $x_0 = 0.5; \quad x_1 = 0.5495737569; \quad x_2 = 0.5775047961; \ldots; \quad x_{44} = 0.6190612867 = x^*$

- The second root seems sufficiently close to $x_0 = 1.5$

  $$\longrightarrow g_1'(1.5) = \frac{1}{3}e^{1.5} = 1.493896357 > 1 \quad ✗$$
  $$\longrightarrow g_2'(1.5) = \frac{1}{1.5} = \frac{2}{3} < 11 \quad ✓$$

  Therefore, to find the second root we iterate using $g_2(x)$:

  $x_0 = 1.5; \quad x_1 = 1.504077397; \quad x_2 = 1.506791973; \ldots; \quad x_{44} = 1.512134552 = x^*$

(iii) **Newton-Raphson.** Let us recall the Newton-Raphson formula:

$$x_{n+1} = x_n - \overbrace{\frac{f(x_n)}{f'(x_n)}}^{=r(x)} = x_n - r(x_n)$$

$$f(x) = e^x - 3x; \quad f'(x) = e^x - 3; \quad r(x) = \frac{f(x)}{f'(x)} = \frac{e^x - 3x}{e^x - 3}$$

Selecting a sufficiently close initial value x_0 for each of the roots:

- 1st root; let $x_0 = 0.5$.
  - Convergence condition:

$$\left|\frac{f(0.5)f''(0.5)}{f'(0.5)^2}\right| = 0.1342859103 < 1 \quad \checkmark$$

  - Thus convergence is guaranteed for $x_0 = 0.5$; iterating:

$$x_0 = 0.5; \quad x_1 = 0.610059655; \quad x_2 = 0.6189967797; \quad x_3 = 0.6190612834;$$
$$x_4 = 0.6190612867 = x^*$$

- 2nd root; let $x_0 = 1.5$.
  - Convergence condition:

$$\left|\frac{f(1.5)f''(1.5)}{f'(1.5)^2}\right| = 0.03737988514 < 1 \quad \checkmark$$

  - Thus convergence is guaranteed for $x_0 = 1.5$; iterating:

$$x_0 = 1.5; \quad x_1 = 1.512358146; \quad x_2 = 1.512134625; \quad x_3 = 1.512134552 = x^*$$

Note: Let me remind you that the convergence conditions of the fixed-point/function iteration method and Newton-Raphson method are sufficient but not necessary. Therefore, you'll see that on the Excel file I have used some x_0 that actually do not fulfil the corresponding convergence condition and, nevertheless, convergence is achieved.