AS1056 - Mathematics for Actuarial Science. Chapter 16, Tutorial 1.

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Refreshing some concepts: Recurrence relations

A **recurrence relation** is an equation that expresses each element of a sequence $\{a_n\}$ as a function of the preceding ones.

$$\boxed{a_n = f\left(n, a_{n-1}, a_{n-2}, \ldots\right), \quad n \in \mathbb{N}^*}$$

A recurrence relation is called a **difference equation** if it takes the form

$$\boxed{a_n + b_1 a_{n-1} + \ldots + b_k a_{n-k} = f(n), \quad n \geq k}$$

for some constant coefficients $b_1, \ldots, b_k$.

Solving Recurrence Relations other than Difference Equations

1. **Linear Recurrence Relations with Variable Coefficients**

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In a case like this we proceed as follows:

$$A_n = \frac{a_n}{b_n b_{n-1} ... b_1} = \overbrace{\frac{\cancel{b_n} a_{n-1}}{\cancel{b_n} b_{n-1} ... b_1}}^{=A_{n-1}} + \frac{c_n}{b_n b_{n-1} ... b_1} = A_{n-1} + C_n$$

The solution of this first-order difference equation is $A_N = A_0 + \sum_{n=1}^{N} C_n$.

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3. **Convergence to a Limit**

Exercise 16.5

Find and expression for the solution a_n of the recurrence relation

$$a_n = n + \frac{1}{n-1} a_{n-1}$$

with initial condition $a_1 = 1$.
Is this a difference equation?

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$\longrightarrow$ **No, since the coefficients are not constant, but depend of n.**

❊ *Linear recurrence relation with variable coefficients.*

Proof by induction

Mathematical induction is a method for proving that some statement $P(n)$ is true for any natural number n.
A **proof by induction** of some statement $P(n)$ consists of two steps:

1. **Base Case**: Prove the statement $P(n)$ for an initial value of n, usually, $n = 0$ or $n = 1$. The purpose of this step is to show that the statement holds for the first number in the sequence of natural numbers.
2. **Induction Step**: Assume that the statement $P(n)$ is true for some arbitrary natural number $n = k$, $n \geq 1$ (induction hypothesis). Finally, prove that, if $P(k)$ is true, then the next case, $P(k + 1)$, also holds.

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Then, if the statement is true for one case (base case) + assuming that the statement holds for an arbitrary case k **implies** it holds for the next case $k+1$, $\implies$ the statement must be true for all $n \in \mathbb{N}$

Think of it as dominoes. The base case is like knocking over the first domino, and the inductive step is what ensures that if one domino falls, the next one will too. And, therefore, all the dominoes (i.e., the natural numbers) will fall.

Exercise 16.6

The sequence $\{a_n : n = 0, 1, 2, \ldots\}$ has initial value $a_0 = 3$ and satisfies the equation $a_n = \frac{n}{n+1} a_{n-1} + 1$.

(i) Work out the values of $a_1, \ldots, a_5$, expressing them as fractions rather than decimals.

(ii) You should notice that the denominator of an is (in most cases) $n + 1$. Define $c_n = (n + 1)a_n$ and write down the values of c_n for n going from 0 to 6.

(iii) Find A, B and C such that $c_n = A + Bn + Cn^2$.

(iv) Formulate your findings as an inductive hypothesis.

(v) Check that the hypothesis is satisfied for $n = 0$.

(vi) Prove the inductive hypothesis.

Proof by Induction