AS1056 - Mathematics for Actuarial Science. Chapter 15, Tutorial 2.

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Refreshing some concepts: Second-order linear ODEs

A **second-order linear differential equation** is one of the form:

$$\boxed{\frac{d^2y(x)}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x)}$$

The general solution of a second-order linear ODE can be written as:

$$y(x) = \underbrace{\eta(x)}_{\text{particular integral}} + \underbrace{y_0(x)}_{\text{complementary function}}$$

where η and y_0 respectively satisfy:

$$\longrightarrow \eta''(x) + a(x)\eta(x)' + b(x)\eta(x) = f(x)$$
$$\longrightarrow y_0''(x) + a(x)y_0'(x) + b(x)y(x) = 0$$

How do we find $\eta(x)$ and $y_0(x)$?

- $\eta(x)$: generally a matter of guesswork, guided by the form of the function f.
- $y_0(x)$: in case of constant coefficients, *auxiliary equation* method.

The **auxiliary equation** method provides a straightforward way to find solutions to the complementary function of a linear ODEs with constant coefficients.

$$\lambda^2 + a\lambda + b = 0$$

which has roots λ_1 and λ_2. Depending on the sign of the discriminant we have:

- If $a^2 - 4b > 0$, $\lambda_1 \neq \lambda_2$ and:

$$\longrightarrow y_0(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

- If $a^2 - 4b < 0$, $\lambda_1 = \mu + i\nu$, $\lambda_2 = \mu - i\nu$ and:

$$\longrightarrow y_0(x) = e^{\mu x}\left(Ae^{i\nu x} + Be^{-i\nu x}\right) = e^{-2x}\left(C\cos(\nu x) + D\sin(\nu x)\right)$$

- If $a^2 - 4b = 0$, $\lambda_1 = \lambda_2 = \mu$ and:

$$\longrightarrow y_0(x) = (Ax + B)\, e^{\mu x}$$

Exercise 15.10

An object is taken up to a height 10 km and is dropped. It accelerates under gravity but is subject to air resistance, which is proportional to the square of the speed. Let $x(t)$ represent the height of the object at time t, $v(t)$ its speed in a downwards direction. Then

$$\frac{dx}{dt} = -v(t), \quad \frac{dv}{dt} = g - cv(t)^2$$

(i) Solve the second equation by partial fractions if $v(0) = 0$.

(ii) Show that $\lim_{t\to\infty} v(t) = \sqrt{g/c}$. (This is the concept of "terminal velocity", a speed when deceleration due to air pressure balances the acceleration due to gravity.)

(iii) Show that $x(t)$, measured in metres, is given by

$$x(t) = 10^4 + \frac{\ln(2)}{c} - \frac{1}{c}\ln\left(e^{t\sqrt{gc}} + e^{-t\sqrt{gc}}\right)$$

(iv) Use the values $g = 10ms^{-2}$, $c = 0.001m^{-1}$ to show that the object hits the ground approximately 100 seconds after it was released.

Exercise 15.8

Two functions $x(t)$ and $y(t)$ satisfy the differential equations $\frac{dx}{dt} = 4xy - x$, $\frac{dy}{dt} = 1 + \ln(x)$.

(i) Define $X(t) = \ln(x(t))$. Rewrite the two DEs in terms of X and y.

$$\begin{cases} \frac{dX}{dt} = \frac{1}{x}\frac{dx}{dt} = 4y - 1 \\ \frac{dy}{dt} = 1 + \ln(e^X) = 1 + X \end{cases}$$

(ii) Obtain a second order DE for X.

$$\frac{d^2X}{dt^2} = 4\frac{dy}{dt} = 4(1+X) \text{ or } \ddot{X} - 4X = 4$$

(iii) Show that $X(t) = -1$ is a particular integral and use this to find the general solution for X and y.

(iv) Evaluate the arbitrary constants in the case where the boundary conditions are $x(0) = 1$, $y(0) = 1$. Hence write down the solutions $x(t)$ and $y(t)$ for all $t \geq 0$.