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# AS1056 - Chapter 15, Tutorial 2. 21-03-2024. Notes.

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**Exercise 15.8**

$$\begin{cases} \dfrac{dx}{dt} = 4xy - x = x(4y-1) \\ \\ \dfrac{dy}{dt} = 1 + \ln(x) \end{cases}$$

(i) Let $X(t) = \ln(x(t))$, then

$\longrightarrow x = e^X$

$\longrightarrow \dfrac{dX}{dt} = \dfrac{1}{x}\dfrac{dx}{dt}$ or $\dot{X} = \dfrac{\dot{x}}{x}$

thus,

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{1}{x}\dfrac{dx}{dt} = 4y - 1 & (1) \\ \dfrac{dy}{dt} = 1 + \ln(e^X) = 1 + X & (2) \end{cases}$$

(ii) To obtain a second-order DE for X we differentiate equation 1 above:

$$\frac{d^2X}{dt^2} = 4\frac{dy}{dt} = 4(1+X) \text{ or } \ddot{X} - 4X = 4$$

(iii)
- Particular Integral

  Replacing $\eta(t) = X(t) = -1$ on the second-order DE of (ii), we see that this is is satisfied:

  $$\frac{d^2X}{dt} = \frac{d^2(-1)}{dt} = 0 = 4(1 + (-1)) = 0$$

- Complementary function (CF)

  To find the CF, we first solve the auxiliary equation:

  $\lambda^2 - 4 = 0; \quad \lambda^2 = 4; \quad \lambda = \pm 2$

  And therefore, the CF is:

  $X_0(t) = Ae^{2t} + Be^{-2t}$

Finally, the general solution to the ODE is:

$$\begin{cases} X(t) = \eta(t) + X_0(t) = -1 + Ae^{2t} + Be^{-2t} \\ \\ y(t) = \dfrac{1+\dot{X}}{4} = \dfrac{1}{4} + \dfrac{1}{4}\left(2Ae^{2t} - 2Be^{-2t}\right) = \dfrac{1}{4} + \dfrac{1}{2}Ae^{2t} - \dfrac{1}{2}Be^{-2t} \end{cases}$$

(iv) Applying the boundary conditions:

- $x(0) = 1 \implies X(0) = \ln((x(0)) = \ln(1) = 0$

and $X(0) = -1 + A + B = 0; \quad A + B = 1$

- $y(0) = \frac{1}{4} + \frac{1}{2}A - \frac{1}{2}B = 1; \quad \frac{1}{2}(A - B) = \frac{3}{4}; \quad A - B = \frac{3}{2}$

$$\underbrace{\hspace{8em}}$$

$2A = \frac{5}{2}; \quad A = \frac{5}{4}$ and $B = -\frac{1}{4}$

Putting all together:

$$\begin{cases} X(t) = -1 + \frac{5}{4}e^{2t} - \frac{1}{4}e^{-2t} \\ \\ y(t) = \frac{1}{4} + \frac{5}{8}e^{2t} + \frac{1}{8}e^{-2t} \end{cases} \implies \begin{cases} x(t) = \exp\left(-1 + \frac{5}{4}e^{2t} - \frac{1}{4}e^{-2t}\right) \\ \\ y(t) = \frac{1}{4} + \frac{5}{8}e^{2t} + \frac{1}{8}e^{-2t} \end{cases}$$

**Exercise 15.10**

$$\frac{dx}{dt} = -v; \quad \frac{dv}{dt} = g - cv^2$$

- Solve $\frac{dv}{dt} = g - cv^2$ using partial fractions.

  Note this is a first-order separable ODE, thus what we need to do is to separate the $t$'s and the $v$'s and then integrate on both sides. First, let us rewrite it as:

$$\frac{dv}{dt} = \underbrace{\left[\frac{g}{c} - v^2\right]}_{=(\sqrt{\frac{g}{c}}+v)(\sqrt{\frac{g}{c}}-v)} \times c$$

  For notational convenience let $\gamma = \sqrt{\frac{g}{c}}$, then,

$$\frac{dv}{dt} = (\gamma + v)(\gamma - v)c \; ; \quad \frac{dv}{(\gamma+v)(\gamma-v)} = cdt$$

  Since integrating the LHS would be a bit difficult, let us re-express $\frac{1}{(\gamma+v)(\gamma-v)}$ using partial fractions:

$$\frac{1}{(\gamma+v)(\gamma-v)} = \frac{A}{\gamma+v} + \frac{B}{\gamma-v} = \frac{A(\gamma-v) + B(\gamma+v)}{(\gamma+v)(\gamma-v)}$$

  Thus,

$$1 = A(\gamma - v) + B(\gamma + v) = \gamma(A+B) + v(B-A)$$
$$\longrightarrow B - A = 0; \quad B = A$$
$$\longrightarrow \gamma(A+B) = 1; \quad \gamma\underbrace{(A+A)}_{2A} = 1; \quad A = \frac{1}{2\gamma} = B$$

  And now we can write

$$\frac{dv}{(\gamma+v)(\gamma-v)} = \left(\frac{A}{\gamma+v} + \frac{B}{\gamma-v}\right)dv = \frac{1}{2\gamma}\left(\frac{1}{\gamma+v} + \frac{1}{\gamma-v}\right)dv = cdt$$

  that is,

$$\left(\frac{1}{\gamma+v} + \frac{1}{\gamma-v}\right)dv = 2\gamma cdt$$

  Now, integrating on both sides we get:

$$\underbrace{\ln(\gamma+v) - \ln(\gamma-v)}_{\ln\left(\frac{\gamma+v}{\gamma-v}\right)} = 2\gamma ct + A; \quad \frac{\gamma+v}{\gamma-v} = e^{2\gamma ct + A}$$
$$\gamma + v = \gamma e^{2\gamma ct+A} - ve^{2\gamma ct+A}; \quad v\left(e^{2\gamma ct+A} + 1\right) = \gamma\left(e^{2\gamma ct+A} - 1\right)$$
$$\longrightarrow v = \gamma\frac{\left(e^{2\gamma ct+A} - 1\right)}{(e^{2\gamma ct+A} + 1)}$$

Finally, applying the boundary condition $v(0) = 0$:

$$v(0) = \gamma \frac{e^A - 1}{e^A + 1} = 0; \quad e^A = 1; \quad A = \ln(1) = 0$$

Therefore,

$$\longrightarrow v(t) = \gamma \frac{e^{2\gamma ct} - 1}{e^{2\gamma ct} + 1} \underset{\times \frac{e^{-2\gamma ct}}{e^{-2\gamma ct}}}{\underset{\uparrow}{=}} \gamma \frac{1 - e^{-2\gamma ct}}{1 + e^{-2\gamma ct}}$$

(ii)

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \gamma \frac{1 - e^{-2\gamma ct}}{1 + e^{-2\gamma ct}} = \gamma$$

where $\lim_{t\to\infty} e^{-2\gamma ct} = 0$.

(iii) Of course that, given what we just got for $v(t)$, we can derive $x(t)$ by integrating on both sides of $dx = -vdt$. However, that's a bit cumbersome integral. Instead, we are asked to show that "the expression we are provided is indeed $x(t)$". For such purposes, we need to check that it fulfils, on the one hand, the boundary condition and, on the other hand, the ODE itself. Note there's a typo on the statement of the exercise, and that it should say instead:

$$x(t) = 10^4 + \frac{\ln(2)}{c} - \frac{1}{c} \ln(e^{\sqrt{gc}t} + e^{-\sqrt{gc}t}) = 10^4 + \frac{\ln(2)}{c} - \frac{1}{c} \ln(e^{\gamma ct} + e^{-\gamma ct})$$

since $\gamma c = \sqrt{g}c \times c = \sqrt{g} \times c \frac{\cancel{c}}{\cancel{\sqrt{c}}} \frac{\sqrt{c}}{\cancel{\sqrt{c}}} = \sqrt{gc}$ The boundary condition is implicitly given by "An object is taken up to a height 10 km", i.e., $x(0) = 10$ km $= 10,000$ m.

- $x(0) = 10^4 + \frac{\ln(2)}{c} - \frac{1}{c}\ln(2) = 10^4 \quad \checkmark$
- $\frac{dx}{dt} = -\frac{1}{c} \frac{\gamma c e^{\gamma ct} - \gamma c e^{-\gamma ct}}{e^{\gamma ct} - e^{-\gamma ct}} \underset{\times \frac{e^{-\gamma ct}}{e^{-\gamma ct}}}{\underset{\uparrow}{=}} -\gamma \frac{1 - e^{-2\gamma ct}}{1 + e^{-2\gamma ct}} = -v(t) \quad \checkmark$

(iv) $g = 10\text{ms}^{-2}$ and $c = 0.001\text{m}^{-1}$; thus, $\gamma c = \sqrt{gc} = \sqrt{0.01} = 0.1$

Then,

$$x(t) = 10^4 + \frac{\ln(2)}{0.001} - \frac{1}{0.001} \ln\left(e^{0.1t} + e^{-0.1t}\right) = 10^4 + 1,000\ln(2) - 1,000\ln\left(e^{0.1t} + e^{-0.1t}\right)$$

The moment at which the object hits the ground is the values of $t$ for which $x(t) = 0$. So, let's try to solve for $t$:

$$10^4 + 1,000\ln(2) - 1,000\ln\left(e^{0.1t} + e^{-0.1t}\right) = 0$$
$$\ln\left(e^{0.1t} + e^{-0.1t}\right) = 10 + \ln(2)$$
$$e^{0.1t} + e^{-0.1t} - e^{10+\ln(2)} = 0$$

However, this equation hasn't got a closed form solution. Instead, it needs to be solved numerically. For such purposes there's a very famous method due to Newton-Raphson.

This tells us that if $f$ satisfies certain assumptions and the initial guess is close to the solution, then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is a better approximation of the root than $x_0$. So, we start with some initial guess $x_0$ and get some $x_1$. Following this logic, we then repeat the process as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently precise value is reached. In our case we have,

$$f(t) = e^{0.1t} + e^{-0.1t} - e^{10+\ln(2)}$$
$$f'(t) = 0.1e^{0.1t} - 0.1e^{-0.1t}$$

Then, using the calculator we can do the following. Let's take as initial guess $t_0 = 100$:

$$\begin{array}{ccccccc}
t_1 & = & t_0 - \frac{f(t_0)}{f'(t_0)} & = & 100 - \frac{e^{0.1\times 100} + e^{-0.1\times 100} - e^{10+\ln(2)}}{0.1e^{0.1\times 100} - 0.1e^{-0.1\times 100}} & = & 110 \\
t_2 & = & t_1 - \frac{f(t_1)}{f'(t_1)} & = & 110 - \frac{e^{0.1\times 110} + e^{-0.1\times 110} - e^{10+\ln(2)}}{0.1e^{0.1\times 110} - 0.1e^{-0.1\times 110}} & = & 107.3575888\ldots \\
t_3 & = & \vdots & = & \vdots & = & 106.940423\ldots \\
t_4 & = & \vdots & = & \vdots & = & 106.9314758\ldots \\
t_5 & = & \vdots & = & \vdots & = & 106.9314718\ldots \\
t_6 & = & \vdots & = & \vdots & = & 106.9314718\ldots \\
\vdots & = & \vdots & = & \vdots & = & \vdots
\end{array}$$

You'll see that from $t_5$ onwards you'll always get 106.9314718 if you continue iterating. That is, Newton-Raphson has converged to a solution $t = 106.9314718$, and you can check that $x(106.9314718) = 0$.

**Note on Section 14.7 - Bernoulli's Equation** A colleague of yours asked for some further clarification on this part of the lecture notes. So, let's see what we can do... An ordinary differential equation is called a Bernoulli differential equation if it is of the form

$$\frac{dy(x)}{dx} + P(x)y(x) = Q(x)y(x)^n \tag{3}$$

where $n$ is a real number.

- When $n = 0$, the differential equation is linear:

  $$\frac{dy(x)}{dx} + P(x)y(x) = Q(x)$$

  and we can use the integrating factor method, for example.

- When $n = 1$, it is separable:

  $$\frac{dy(x)}{dx} + P(x)y(x) = Q(x)y(x); \quad \frac{dy(x)}{dx} = (Q(x) - P(x))y(x); \quad \frac{dy(x)}{y(x)} = (Q(x) - P(x))dx$$

  and we can just integrate on both sides to solve.

- For $n \neq 0$ and $n \neq 1$, the substitution $u = y(x)^{1-n}$ reduces any Bernoulli equation to a linear differential equation. Note,

  $$u = y(x)^{1-n} \implies y(x) = u^{\frac{1}{1-n}}$$
  $$\implies dy(x) = \frac{1}{1-n}u^{\frac{1}{1-n}-1}du = \frac{1}{1-n}u^{\frac{n}{1-n}}du$$

  then, replacing on equation 3:

  $$\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx} + P(x)u^{\frac{1}{1-n}} = Q(x)u^{\frac{n}{1-n}}$$

  Finally, multiplying on both sides by $(1-n)u^{-\frac{n}{1-n}}$:

  $$\frac{1}{\cancel{1-n}}\cancel{u^{\frac{n}{1-n}}}\frac{du}{dx} \times \cancel{(1-n)}\frac{1}{\cancel{u^{\frac{n}{1-n}}}} + \underbrace{P(x)u^{\frac{1}{1-n}} \times (1-n)\frac{1}{u^{\frac{n}{1-n}}}}_{=(1-n)P(x)u^{\cancel{\frac{1-n}{1-n}}}} = Q(x)\cancel{u^{\frac{n}{1-n}}} \times (1-n)\frac{1}{\cancel{u^{\frac{n}{1-n}}}}$$

  Therefore,

  $$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

  which is a first-order linear ODE with unknown variable $u$, which we can solve by the integrating factor method. Note that, if $P(x) = 0$, then it becomes a separable ODE, and hence we would just need to separate things and integrate on both sides. This is the case in example 14.11:

  $$\frac{dy}{dx} = 1 + \sqrt{y - x}; \quad x \geq 0$$

  Let $u = \sqrt{y - x}$. That is, $u = y(x)^{1-n}$, with $y(x) = y - x$ and $n = \frac{1}{2}$. Then, $y = u^2 + x$ and $dy = 2udu + dx$. Replacing in the ODE:

  $$\frac{2udu + dx}{dx} = 1 + \sqrt{u^2 + \cancel{x} - \cancel{x}}; \quad 2u\frac{du}{dx} + \cancel{1} = \cancel{1} + \sqrt{u^2}$$
  $$2\cancel{u}\frac{du}{dx} = \cancel{u}; \quad 2du = dx$$

  And now integrate on both sides and undo the substitution $u = \sqrt{y - x}$.