AS1056 - Mathematics for Actuarial Science. Chapter 14, Tutorial 1.

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Refreshing some concepts: First order linear ODEs

A first order linear differential equation is one of the form:

$$a_1(x)\frac{dy(x)}{dx} + a_0(x)y(x) = f(x) \quad \text{or} \quad a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

Which, letting $a(x) = \frac{a_0(x)}{a_1(x)}$ and $b(x) = \frac{f(x)}{a_1(x)}$, can also be written as:

$$\frac{dy(x)}{dx} + a(x)y(x) = b(x) \quad \text{or} \quad y'(x) + a(x)y(x) = b(x)$$

Solving procedures:

1. **Integrating factor method**
2. **Complementary function and particular integral**

1. Integrating factor method

Consider the first order linear ODE:

$$\frac{dy(x)}{dx} + a(x)y(x) = b(x)$$

The integrating factor $I(x)$ is given by:

$$I(x) = \exp\left(\int a(x)dx\right); \quad \frac{dI(x)}{dx} = a(x)\underbrace{\exp\left(\int a(x)dx\right)}_{=I(x)}$$

Thus multiplying on both sides of the ODE by $I(x)$:

$$I(x)\left(\frac{dy(x)}{dx} + a(x)y(x)\right) = I(x)b(x) \implies I(x)\frac{dy(x)}{dx} + \underbrace{a(x)I(x)}_{=\frac{dI(x)}{dx}}y(x) = I(x)b(x)$$

And, by the product rule,

$$I(x)\frac{dy(x)}{dx} + y(x)\frac{dI(x)}{dx} = \frac{d}{dx}\left(I(x)y(x)\right) = I(x)b(x)$$

2. Complementary Function and Particular Integral

Consider the first order linear ODE:

$$\frac{dy(x)}{dx} + a(x)y(x) = b(x)$$

The general solution of a first order linear ODE can be written as:

$$y(x) = \underbrace{\eta(x)}_{\text{particular integral}} + \underbrace{y_0(x)}_{\text{complementary function}}$$

where η and y_0 respectively satisfy:

$\eta'(x) + a(x)\eta(x) = b(x)$, and,
$y_0'(x) + a(x)y_0(x) = 0$

... and this works due to the Superposition Principle

Theorem (Superposition Principle)

If y_1 is a solution to the equation

$$ay'' + by' + cy = f_1(t),$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(t),$$

then for any constants k_1 and k_2, the function $k_1 y_1 + k_2 y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1 f_1(t) + k_2 f_2(t).$$

Thus, by superposition principle, the general solution to a nonhomogeneous equation is the sum of the general solution to the homogeneous equation —**complementary function**, $y_0(x)$—, and one particular solution —**particular integral**, $\eta(x)$).

Exercise 14.3/14.4 (ii)

(a) Look for a solution to the equation

$$(1+x)\frac{d\eta(x)}{dx} + x\eta(x) = 2(1+x)^2$$

Solve using the integration factor method.

(b) Find the general solution $y_0(x)$ to $(1+x)\frac{dy_0}{dx} + xy_0(x) = 0$.

(c) Determine the solution to the ODE:

$$(1+x)\frac{dy(x)}{dx} + xy(x) = 2(1+x)^2$$

satisfying the boundary condition $y'(1) = 0$.

Note:

- When it comes to first-order linear ODEs, finding the particular integral essentially means solving the entire differential equation.
- The CF and PI method is more traditionally applied to second-order (or higher) linear ODEs, and its utility and rationale become clearer in that context.

Exercise 14.10

In this exercise we look at one aspect of the Lotka-Volterra predator-prey model. Let x(t) denote the number of rabbits in a population and y(t) the number of foxes. The first Lotka-Volterra equation states that

$$\frac{dx}{dt} = \alpha x(t) - \beta x(t)y(t),$$

where α is the population growth rate in the absence of foxes and β is the rate at which a fox consumes rabbits.

(i) Assume that the number of foxes is kept constant, so that $y(t) = y_0$. If the initial rabbit population is $x(0) = 100$, solve the DE to find $x(t)$ for all t.

(ii) Now assume that the number of foxes grows slowly over time, $y(t) = y_0 e^{0.01t}$. How does that change the evolution of the rabbit population?