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AS1056 - Chapter 14, Tutorial 1. 13-03-2024. Notes.

Exercise 14.3/14.4 (ii)
Determine the solution to the ODE:

$$(1+x)\frac{dy(x)}{dx} + xy(x) = 2(1+x)^2$$

satisfying the boundary condition $y'(1) = 0$. The way the exercise is posed suggests we should solve it using the *Complementary function and Particular Integral* method. The latter involves finding out:

- The particular integral $\eta(x)$, which satisfies $\eta'(x) + a(x)\eta(x) = b(x)$
- The complementary function $y_0(x)$, which satisfies $y_0'(x) + a(x)y_0(x) = 0$

For instance, let's try to find out the particular integral $\eta(x)$, but instead of using the hint provided in the lecture notes, we'll implement the *integrating factor* method.

(a) We want to find a function $\eta(x)$ which satisfies the differential equation:

$$(1+x)\frac{d\eta(x)}{dx} + x\eta(x) = 2(1+x)^2$$

Let us rewrite this first order linear ODE as:

$$\frac{d\eta(x)}{dx} + \underbrace{\frac{x}{1+x}}_{=a(x)}\eta(x) = \underbrace{2(1+x)}_{=b(x)}$$

The integrating factor is:

$$I(x) = \exp\left(\int \frac{x}{1+x}dx\right)$$

To solve the above integral let us substitute $u = 1+x$:

$$\int \frac{x}{1+x}dx = \int \frac{u-1}{u}du = \int 1du - \int \frac{1}{u}du = u - \ln(u) + C = 1 + x - \ln(1+x) + C$$

hence,

$$I(x) = e^{1+x-\ln(1+x)+C} = \frac{e^{1+x+C}}{1+x}$$

Multiplying by $I(x)$ on both sides of the ODE, we know we get:

$$\frac{d}{dx}\left(I(x)\eta(x)\right) = I(x)b(x)$$

that is,

$$\frac{d}{dx}\left(\frac{e^{1+x}}{1+x}\eta(x)\right) = \frac{e^{1+x}}{\cancel{1+x}}2\cancel{(1+x)} = 2e^{1+x}$$

Integrating both sides from 0 to x:

$$\frac{e^{1+x}}{1+x}\eta(x) - e\eta(0) = 2e\int_0^x e^x dx = 2e(e^x - 1)$$

$$\frac{e^{1+x}}{1+x}\eta(x) = 2e^{1+x} \underbrace{-2e + e\eta(0)}_{e(\eta(0)-2)}$$

$$\eta(x) = 2(1+x) + e^{-x}(1+x)(\eta(0)-2)$$

Now, you may have already noticed that what we've just found out is actually the general solution to the ODE. Indeed, finding the general solution for $\eta(x)$ is tantamount to solving the differential equation itself. On the other hand, if we follow the hint provided in the lecture notes and use a guess for $\eta(x)$, we will arrive at a particular solution for $\eta(x)$. In such a case, we must continue with the steps of the Complementary Function (CF) and Particular Integral (PI) method to find the general solution of the ODE.

The method of CF and PI is traditionally applied to second-order (or higher) linear ODEs, and its utility and rationale become clearer in that context. This is why it is crucial to become familiar with this method from the very beginning, even if applying it to first-order linear ODEs might seem a bit nonsense —especially when the integrating factor method presents a much more straightforward alternative.

Applying the boundary condition $\eta'(1) = 0$:

$$\eta'(x) = 2 + (\eta(0)-2)\left(-e^{-x}(1+x) + e^{-x}\right) = 2 + e^{-x}\left(\eta(0)-2\right)(-\cancel{1} - x + \cancel{1}) =$$
$$= 2 - xe^{-x}(\eta(0)-2)$$

Then,

$$\eta'(1) = 2 - e^{-1}\left(\eta(0)-2\right) = 0; \quad \eta(0) - 2 = 2e; \quad \eta(0) = 2e+2$$

Finally,

$$\longrightarrow \eta(x) = y(x) = 2(1+x) + e^{-x}(1+x)(2e + \cancel{2} - \cancel{2}) = 2(1+x)(1+e^{1-x})$$

The exercise would be already solved. However —and in order to remain as faithful as possible to what the lecture notes' statement of the exercise tells us—, we could have, alternatively, proceeded as follows. Note that the hint suggests finding a particular integral of the form $\eta(x) = (A+Bx)$. Given the general solution we have just obtained for $\eta(x)$, this implicitly tells us that we need to impose the boundary condition $\eta(0) = 2$, in order to get to the particular solution "the hint want us to get to". That would be $\eta(x) = 2(1+x)$. Please check if you arrive to the same result by setting $\eta(x) = A + Bx$ and replacing on the ODE, as the lecture notes suggest.

(b) The second step is to find the complementary function, that is, the solution to the homogeneous equation associated with the original differential equation. So we need to solve for $y_0(x)$ the following:

$$(1+x)y_0'(x) + xy_0(x) = 0$$

Then,

$$y_0'(x) = -\frac{x}{(1+x)}y_0(x); \qquad \underbrace{\frac{y_0'(x)}{y_0(x)}}_{=\frac{d}{dx}\ln(y_0(x))} = -\frac{x}{1+x}$$

Integrating on both sides w.r.t x,

$$\int \frac{d}{dx}\ln(y_0(x))dx = -\int \frac{x}{1+x}dx;$$
$$\ln(y_0(x)) = -\int \frac{1+x-(1+x)+x}{1+x}dx = -\int 1 - \frac{1}{1+x}dx = -x + \ln(1+x) + C$$

Finally, we get:

$$\longrightarrow y_0(x) = (1+x)e^{C-x}$$

(c) Now that we have both the particular integral and the complementary function, we can say that, by the superposition principle theorem (see tutorial slides), the general solution of the given ODE is:

$$y(x) = \eta(x) + y_0(x) = 2(1+x) + (1+x)e^{C-x}$$

And now we can apply the boundary condition $y'(0) = 1$:

$$y'(x) = 2 + e^{C-x} - (1+x)e^{C-x} = 2 + e^{C-x}(\not{1} - \not{1} - x) = 2 - xe^{C-x}$$

Thus,

$$y'(1) = 2 - xe^{C-1} = 0; \quad 2 = e^{C-1}; \quad e^C = 2e$$

Finally, the particular solution of the ODE is:

$$y(x) = \eta(x) + y_0(x) = 2(1+x) + (1+x)e^{1-x} = 2(1+x)(1+e^{1-x})$$

which is the same we got on (a) solving directly with the integrating factor method.

Exercise 14.10

$$\frac{d(x(t)}{dt} = \alpha x(t) - \beta x(t)y(t)$$

(ii) $x(0) = 100$; $y(t) = y_0 e^{0.01t}$

The first Lotka-Voltera equation is a separable ODE and thus we can easily operate,

$$\frac{dx(t)}{dt} = x(t)\left(\alpha - \beta y(t)\right); \quad \frac{dx(t)}{x(t)} = \left(\alpha - \beta y(t)\right)dt$$

Integration both sides between 0 and t:

$$\int_0^t \frac{dx(t)}{x(t)} = \int_0^t \left(\alpha - \beta y(t)\right)dt$$

then,

$$\ln(x(t)) - \underbrace{\ln(x(0))}_{\ln(100)} = \int_0^t \left(\alpha - \beta y_0 e^{0.01t}\right) dt = \alpha \int_0^t 1dt - \beta y_0 \int_0^t e^{0.01t} dt =$$

$$= \alpha \left[t\right]_0^t - \beta y_0 \left[\frac{e^{0.01t}}{0.01}\right]_0^t = \alpha t - 100\beta y_0 \left(e^{0.01t} - 1\right)$$

Finally,

$$x(t) = 100 \exp\left(\alpha t - 100\beta y_0 \left(e^{0.01t} - 1\right)\right)$$

Now take the exponent of this expression:

$$\alpha t - 100\beta y_0 \left(e^{0.01t} - 1\right)$$

Let us find the maximum of this expression. Taking first derivatives and equating to zero:

$$\alpha - 100\beta y_0 0.01 e^{0.01t} = 0$$
$$\alpha = \beta y_0 e^{0.01t}; \quad e^{0.01t} = \frac{\alpha}{\beta y_0}; \quad t^* = 100 \ln\left(\frac{\alpha}{\beta y_0}\right)$$

And checking that the second derivative is negative:

$$-0.01\beta y_0 e^{0.01t} < 0 \; \forall t$$

we confirm that t^* is a maximum. We conclude that $\frac{dx(t)}{dt} > 0$ for $t < t^*$ and that $\frac{dx(t)}{dt} < 0$ for $t > t^*$. In other words, the population of rabbits increases up to t^* and then it decreases from t^* onwards. Note that this holds only if $\alpha > \beta y_0$, since otherwise $t^* < 0$, which would imply that the population of rabbits is decreasing from $t = 0$.