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Exercise 13.5

(i) Let us start by simplifying the Cartesian expression of the point z that we were provided:

$$z = \frac{(1+i)(2+i)}{3-i} = \frac{2+i+2i-1}{3-i} = \frac{(1+3i)(3+i)}{9+1} = \frac{3+i+9i-3}{10} = \frac{10i}{10} = i$$

Then,

- $r = |z| = |i| = \sqrt{0+1} = 1$
- $z = r\underbrace{\cos(\theta)}_{=0} + ir\underbrace{\sin(\theta)}_{=1}; \quad \arg(z)\theta = \frac{\pi}{2}$

hence, the polar expression of z is:

$$z = i = i\sin\left(\frac{\pi}{2}\right) = e^{i\frac{\pi}{2}}$$

(ii) We are given a point z expressed in Cartesian form. Let me start calling $z_1 = 2+2i$ and $z_2 = 2-2i$, that is:

$$z = \sqrt{2+2i} - \sqrt{2-2i} = \sqrt{z_1} - \sqrt{z_2}$$

Note that z_1 and z_2 are conjugates of each other. Let us start working with $z_1 = 2+2i$, and noticing that:

- the complex number $z_1 = 2+2i$ is represented as the point $(2,2)$ in the Argand diagram;
- this point is located at a distance of $r = |z_1| = |2+2i|$ from the origin, i.e.,

  $$r = |z_1| = |2+2i| = \sqrt{2^2+2^2} = \sqrt{8}.$$

- And, since $\tan(\theta) = \frac{\text{coefficient of imaginary part}}{\text{coefficient of real part}} = \frac{2}{2}$, i.e., $\arg(z_1) = \theta_{z_1} = \frac{\pi}{4}$, we know that the angle z_1 describes with respect to the x-axis of the Argand diagram is $\frac{\pi}{4} = 45°$.

Hence, z_1 can be expressed in polar form as:

$$z_1 = 2+2i = \sqrt{8}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \underbrace{\sqrt{8}}_{=(2^3)^{1/2}} e^{i\frac{\pi}{4}} = 2^{3/2}e^{i\frac{\pi}{4}}; \quad \sqrt{z_1} = \sqrt{2+2i} = 2^{3/4}e^{i\frac{\pi}{8}}$$

Similarly, given that $z_2 = z_1^*$ is the complex conjugate of z_1, we can easily derive $\sqrt{z_2}$:

$$\sqrt{z_2} = \sqrt{2-2i} = 2^{3/4}e^{-i\frac{\pi}{8}}.$$

Note that both for $\sqrt{z_1}$ and $\sqrt{z_2}$, I just took the *principal* square root (check section 13.5.4 of the lecture notes).

Working on z we now get:

$$z = \sqrt{2+2i} - \sqrt{2-2i} = 2^{3/4}\Bigg[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right) - \underbrace{\cos\left(-\frac{\pi}{8}\right)}_{=\cos\left(\frac{\pi}{8}\right)} - i\underbrace{\sin\left(-\frac{\pi}{8}\right)}_{=-\sin\left(\frac{\pi}{8}\right)}\Bigg] =$$

$$= 2^{3/4} \times 2 \times i\sin\left(\frac{\pi}{8}\right) = i2^{7/4}\sin\left(\frac{\pi}{8}\right) = i2^{7/4}\frac{1}{2}\sqrt{2-\sqrt{2}} = i2^{3/4}\sqrt{\sqrt{2}(\sqrt{2}-1)} =$$

$$= i2^{3/4}2^{1/4}\sqrt{\sqrt{2}-1} = i2\sqrt{\sqrt{2}-1}$$

which is is a purely imaginary positive number, and where $\sin\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{\sqrt{2}-1}$. Finally,

- $r = |z| = |\sqrt{2+2i} - \sqrt{2-2i}| = |i2^{7/4}\overbrace{\sin\left(\frac{\pi}{8}\right)}^{>0}| \underset{\substack{\uparrow \\ |i| = \sqrt{0^2+1^2} = 1}}{=} 2^{7/4}\sin\left(\frac{\pi}{8}\right) = 2\sqrt{\sqrt{2}-1}$
- $z = r\underbrace{\cos(\theta)}_{=0} + ir\underbrace{\sin(\theta)}_{=1}; \quad \theta = \frac{\pi}{2}$

Therefore, z can be expressed in polar form as:

$$z = 2\sqrt{\sqrt{2}-1} \times e^{i\frac{\pi}{2}}.$$

$\sin\left(\frac{\pi}{8}\right)$? Let us recall the double angle formula for the cosine:

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

Then for $\theta = \frac{\pi}{8}$ we have,

$$2\sin^2\left(\frac{\pi}{8}\right) = 1 - \underbrace{\cos\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{2}$$

$$2^2\sin^2\left(\frac{\pi}{8}\right) = 2\sqrt{2}; \quad 2\sin\left(\frac{\pi}{8}\right) = \sqrt{2-\sqrt{2}}; \quad \sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$\cos\left(\frac{\pi}{8}\right)$? (We will use this in the alternative solution as follows) The double angle formula for the cosine also tells us that:

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

Then for $\theta = \frac{\pi}{8}$ we have,

$$\underbrace{\cos\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}} = 2\cos^2\left(\frac{\pi}{8}\right) - 1; \quad \frac{\sqrt{2}+2}{2} = 2\cos^2\left(\frac{\pi}{8}\right);$$

$$2+\sqrt{2} = 2^2\cos^2\left(\frac{\pi}{8}\right); \quad \sqrt{2+\sqrt{2}} = 2\cos\left(\frac{\pi}{8}\right); \quad \cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Alternative solution:

One of your colleagues suggested and alternative approach to this exercise, which I think is quite interesting to explore and therefore I present as follows. Start noting that we can write $x + iy = \sqrt{2+2i}$. Then,

$$z = \sqrt{2+2i} - \sqrt{2-2i} = x + iy - (x - iy) = 2iy \tag{1}$$

For instance, let us focus in $x + iy = \sqrt{2+2i}$:

$$x + iy = \sqrt{2+2i}; \quad (x+iy)^2 = 2+2i; \quad x^2 + 2xiy + i^2y^2 = x^2 + 2xiy - y^2 = 2 + 2i$$

Then,

$$\begin{cases} x^2 - y^2 = 2, \\ 2xy = 2 \implies x = \dfrac{1}{y} \end{cases}$$

Note that I have written $x = \frac{1}{y}$ (and not $y = \frac{1}{x}$) because we are actually only interested in solving for y as Equation 1 is telling us. Then, replacing $x = \frac{1}{y}$ on $x^2 - y^2 = 2$:

$$\left(\frac{1}{y}\right)^2 - y^2 = 2; \quad \frac{1-y^4}{y^2} = 2; \quad -y^4 - 2y^2 + 1 = 0$$

Replace $u = y^2$ and solve:

$$-u^2 - 2u + 1 = 0$$

$$u = \frac{2 \pm \sqrt{4 - 4 \times (-1) \times 1}}{2 \times (-1)} = \begin{cases} u_1 = -1 - \dfrac{\sqrt{8}}{2} = -1 - \sqrt{2} \\ u_2 = -1 + \dfrac{\sqrt{8}}{2} = -1 + \sqrt{2} \end{cases}$$

Replacing back, u_1 yields the following (imaginary) solutions for y:

$$\begin{cases} y_1 = \sqrt{\underbrace{-1 - \sqrt{2}}_{<0}} = \sqrt{(-1)(1 + \sqrt{2})} = i\sqrt{\sqrt{2} + 1} \\ y_2 = -\sqrt{\underbrace{-1 - \sqrt{2}}_{<0}} = -i\sqrt{\sqrt{2} + 1} \end{cases}$$

On the other hand, u_2 yields the following (real) solutions for y:

$$\begin{cases} y_3 = \sqrt{\sqrt{2} - 1} \\ y_4 = -\sqrt{\sqrt{2} - 1} \end{cases}$$

Finally, we get the following values for $z = 2iy$, respectively:

$$\begin{cases} z^{(i)} = -2\sqrt{\sqrt{2} + 1}; \quad r_1 = |z^{(i)}| = \left|-2\sqrt{\sqrt{2} + 1}\right| = 2\sqrt{\sqrt{2} + 1}; \quad \cos(\theta_1) = -1 \implies \theta_1 = \pi \\ z^{(ii)} = 2\sqrt{\sqrt{2} + 1}; \quad r_2 = |z_2| = \left|2\sqrt{\sqrt{2} + 1}\right| = 2\sqrt{\sqrt{2} + 1}; \quad \cos(\theta_2) = 1 \implies \theta_2 = 0 \\ z^{(iii)} = i2\sqrt{\sqrt{2} - 1}; \quad r_3 = |z^{(iii)}| = \left|i2\sqrt{\sqrt{2} - 1}\right| = 2\sqrt{\sqrt{2} - 1}; \quad \sin(\theta_3) = 1 \implies \theta_3 = \dfrac{\pi}{2} \\ z^{(iv)} = -i2\sqrt{\sqrt{2} - 1}; \quad r_4 = |z^{(iv)}| = \left|-i2\sqrt{\sqrt{2} - 1}\right| = 2\sqrt{\sqrt{2} - 1}; \quad \sin(\theta_4) = -1 \implies \theta_4 = \dfrac{3\pi}{2} \end{cases}$$

And their corresponding polar representations are:

$$\begin{cases} z^{(i)} = -2\sqrt{\sqrt{2} + 1} = 2\sqrt{\sqrt{2} + 1} \times e^{i\pi} \\ z^{(ii)} = 2\sqrt{\sqrt{2} + 1} = 2\sqrt{\sqrt{2} + 1} \times e^{i \times 0} \\ z^{(iii)} = i2\sqrt{\sqrt{2} - 1} = 2\sqrt{\sqrt{2} - 1} \times e^{i\frac{\pi}{2}} \\ z^{(iv)} = -i2\sqrt{\sqrt{2} - 1} = 2\sqrt{\sqrt{2} - 1} \times e^{i\frac{3\pi}{2}} \end{cases}$$

Recall that every complex number has two square roots. Therefore, for $z = \sqrt{2 + 2i} - \sqrt{2 - 2i}$, since each square root —$\sqrt{2 + 2i}$ and $\sqrt{2 - 2i}$— can take two different values, there are four possible combinations for these roots. Consequently, z can take up to four distinct values, each with its own corresponding polar representation. But, why did we obtain only one polar representation of z with the first approach to solve the exercise? The reason is that, on the first approach, when taking the square roots $\sqrt{z_1}$ and $\sqrt{z_2}$, we selected only the *principal* square roots (check section 13.5.4 of the lecture notes). If we had considered the secondary square roots as well, we would have also derived four different values for z, each with its corresponding polar representation, by exploring the different combinations. And these should coincide with the polar expressions we have obtained through the second approach.

Interestingly, it can be observed that $z^{(i)}$ and $z^{(ii)}$ are reflections of each other across the imaginary axis of the Argand diagram, indicating they have the same magnitude but opposite signs for their real parts. Similarly, $z^{(iii)}$ and $z^{(iv)}$ are reflections of each other across the real axis of the Argand diagram, suggesting they have the same magnitude but opposite signs for their imaginary parts. However, it's important to recognise that while pairs are reflections of each other, all four representations —$z^{(i)}$, $z^{(ii)}$, $z^{(iii)}$ and $z^{(iv)}$— constitute distinct polar representations of the complex number z, each with unique positions and interpretations in the Argand diagram.

I think that a limitation of this second approach we have implemented might be the inability to distinguish which specific z results from taking the principal square root of $\sqrt{2+2i}$ and the principal square root of $\sqrt{2-2i}$. Perhaps you can figure out some way to detect this distinction...

For the sake of completeness, let us check that if on the first approach we had also considered the secondary square roots we would also get four different polar representations that coincide with the polar representations obtained through the second approach. So, let us start writing the square roots of z_1 and z_2:

$$\sqrt{z_1} = \begin{cases} 2^{3/4} e^{i\frac{\pi}{8}} \\ 2^{3/4} e^{i(\pi+\frac{\pi}{8})} = 2^{3/4} e^{i\frac{9}{8}\pi} \end{cases} \qquad \sqrt{z_2} = \begin{cases} 2^{3/4} e^{-i\frac{\pi}{8}} \\ 2^{3/4} e^{i(\pi-\frac{\pi}{8})} = 2^{3/4} e^{i\frac{7}{8}\pi} \end{cases}$$

Let me recall that $\cos(\pi-\theta) = -\cos(\theta)$ and $\sin(\pi-\theta) = \sin(\theta)$ and consider now the following possible combinations:

- $\sqrt{z_1} = 2^{3/4} e^{i\frac{9}{8}\pi}$ and $\sqrt{z_2} = 2^{3/4} e^{-i\frac{\pi}{8}}$:

  $$z = 2^{3/4}\left[\cos\left(\frac{9}{8}\pi\right) + i\sin\left(\frac{9}{8}\pi\right) - \cos\left(-\frac{\pi}{8}\right) - i\sin\left(-\frac{\pi}{8}\right)\right].$$

  Note that,

  $$\cos\left(\frac{9}{8}\pi\right) = \cos\left(\pi - \left(-\frac{\pi}{8}\right)\right) = -\cos\left(-\frac{\pi}{8}\right) = -\cos\left(\frac{\pi}{8}\right)$$
  $$\sin\left(\frac{9}{8}\pi\right) = \sin\left(\pi - \left(-\frac{\pi}{8}\right)\right) = \sin\left(-\frac{\pi}{8}\right) = -\sin\left(\frac{\pi}{8}\right)$$

  therefore,

  $$z = 2^{3/4}\left[-\cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right) - \cos\left(-\frac{\pi}{8}\right) - i\sin\left(-\frac{\pi}{8}\right)\right] = -2^{7/4}\cos\left(\frac{\pi}{8}\right) =$$
  $$= -2^{7/4}\frac{\sqrt{2+\sqrt{2}}}{2} = -2^{3/4}\sqrt{\sqrt{2}(\sqrt{2}+1)} = -2\sqrt{\sqrt{2}+1} = z^{(i)}$$

- $\sqrt{z_1} = 2^{3/4} e^{i\frac{\pi}{8}}$ and $\sqrt{z_2} = 2^{3/4} e^{i\frac{7}{8}\pi}$:

  $$z = 2^{3/4}\left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right) - \cos\left(\frac{7}{8}\pi\right) - i\sin\left(\frac{7}{8}\pi\right)\right]$$

Note that,

$$\cos\left(\frac{7}{8}\pi\right) = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\left(\frac{\pi}{8}\right)$$
$$\sin\left(\frac{7}{8}\pi\right) = \sin\left(\pi - \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right)$$

therefore,

$$z = 2^{7/4}\cos\left(\frac{\pi}{8}\right) = 2^{7/4}\frac{\sqrt{2+\sqrt{2}}}{2} = 2^{3/4}\sqrt{\sqrt{2}(\sqrt{2}+1)} = 2\sqrt{\sqrt{2}+1} = z^{(ii)}$$

- $\sqrt{z_1} = 2^{3/4}e^{i\frac{\pi}{8}}$ and $\sqrt{z_2} = 2^{3/4}e^{-i\frac{\pi}{8}}$, which we already know that can be simplified to

  $$z = i2^{7/4}\sin\left(\frac{\pi}{8}\right) = i2\sqrt{\sqrt{2}-1} = z^{(iii)}.$$

- $\sqrt{z_1} = 2^{3/4}e^{i\frac{9}{8}\pi}$ and $\sqrt{z_2} = 2^{3/4}e^{i\frac{7}{8}\pi}$:

  $$z = 2^{3/4}\left[\cos\left(\frac{9}{8}\pi\right) + i\sin\left(\frac{9}{8}\pi\right) - \cos\left(\frac{7}{8}\pi\right) - i\sin\left(\frac{7}{8}\pi\right)\right]$$

  Note that,

  $$\cos\left(\frac{9}{8}\pi\right) = -\cos\left(\frac{\pi}{8}\right) = -\cos\left(\pi - \frac{7}{8}\pi\right) = \cos\left(\frac{7}{8}\pi\right)$$
  $$\sin\left(\frac{9}{8}\pi\right) = -\sin\left(\frac{\pi}{8}\right) = -\sin\left(\pi - \frac{7}{8}\pi\right) = -\sin\left(\frac{7}{8}\pi\right)$$

  therefore,

  $$z = 2^{3/4}\left[\cos\left(\frac{7}{8}\pi\right) - i\sin\left(\frac{7}{8}\pi\right) - \cos\left(\frac{7}{8}\pi\right) - i\sin\left(\frac{7}{8}\pi\right)\right] = -i2^{7/4}\sin\left(\frac{7}{8}\pi\right) =$$
  $$= -i2^{7/4}\sin\left(\frac{\pi}{8}\right) = -i2^{7/4}\frac{\sqrt{2-\sqrt{2}}}{2} = -i2^{3/4}\sqrt{\sqrt{2}(\sqrt{2}-1)} = -i2\sqrt{\sqrt{2}-1} = z^{(iv)}$$

Exercise 13.9

$$p(x) = x^4 - 8x^3 + 33x^2 - 68x + 52$$

We are told that the first root is $x = 2 + 3i$. By the hint, we know that the second root is just the conjugate of the latter $x = 2 - 3i$. To find the other two roots we can then divide $p(x)$ by:

$$(x - 2 - 3i)(x - 2 + 3i) = x^2 - 2x + \cancel{3ix} - 2x + 4 - \cancel{6i} - \cancel{3ix} + \cancel{6i} + 9 = x^2 - 4x + 13$$

Let me remind you the steps of polynomial long division:

1. Divide the highest-order term of the numerator by the highest-order term of the denominator, and put that in the answer.
2. Multiply the denominator by that answer, put that below the numerator. Subtract to create a new polynomial.

3. Repeat the process taking now the latter polynomial as the numerator.
4. Stop when the remainder is of a lower degree than the denominator or when it becomes zero.

$$
\begin{array}{r|l}
 & x^2 - 4x + 4 \\
\hline
x^2 - 4x + 13 & x^4 - 8x^3 + 33x^2 - 68x + 52 \\
 & -(x^4 - 4x^3 + 13x^2) \\
\hline
 & 4x^3 + 20x^2 - 68x + 52 \\
 & -(4x^3 - 16x^2 + 52x) \\
\hline
 & 4x^2 - 16x + 52 \\
 & -(4x^2 - 16x + 52) \\
\hline
 & 0
\end{array}
$$

Hence,

$$p(x) = (x^2 - 4 + 13)\underbrace{(x^2 - 4x + 4)}_{(x-2)^2}$$

And therefore, the third/fourth root is $x = 2$.