AS1056 - Mathematics for Actuarial Science. Chapter 12, Tutorial 2.

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Refreshing some concepts: Eigenvalues an eigenvectors

Let A be a linear transformation represented by a matrix. If there is a vector $\mathbf{v} \in \mathbb{R}^n \neq \mathbf{0}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

for some scalar λ, then λ is called the **eigenvalue** of A with corresponding **eigenvector** $\mathbf{v}$.

From a geometric perspective:

- A can be thought as a linear transformation: $\mathbf{u} = A\mathbf{v}$.
- An eigenvector of matrix A is a non-zero vector that, under the linear transformation applied by A, is only scaled (or flipped) by a scalar factor, λ.
- The eigenvalue λ tells you how much the eigenvector $\mathbf{v}$ is stretched or shrunk after the transformation A. If $\lambda > 1$, the vector is stretched; if $0 < \lambda < 1$, it is shrunk. If $\lambda < 0$, the vector is also flipped in direction.

Exercise 12.10

(i) Construct a 2×2 matrix A with all its elements non-zero and having eigenvalues 3 and -1.

(ii) The *characteristic equation* of the matrix A is the polynomial satisfied by the eigenvalues, i.e.,

$$\lambda^2 - 2\lambda - 3 = 0.$$

Show that

$$A^2 - 2A - 3I = \mathbf{O},$$

where $\mathbf{O}$ is a matrix with every element equal to 0.

(This is a special case of the "Cayley-Hamilton Theorem".)

Cayley–Hamilton Theorem

Let A be an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

The Cayley–Hamilton theorem asserts that the matrix A satisfies its own characteristic polynomial, that is,

$$p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = \mathbf{O}$$

where $\mathbf{O}$ is the $n \times n$ zero matrix.

Exercise 12.12

The function $f(\theta)$, defined on $\theta \in [0, \pi]$, is given by the quadratic form

$$f(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \underbrace{\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

Theory suggests that the **maximum value** of f over the given range should be equal to the **larger eigenvalue** of matrix A and the **minimum value** should be equal to the **smaller eigenvalue**.

(i) Show that one of the turning points of f is $\theta = \frac{\pi}{8}$ and that it is a maximum.

Exercise 12.12

Hint. Recall the following trigonometric identities,

- Pythagorean formula for sines and cosines:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

- Double angle formulas for sine and cosine:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Exercise 12.12

(ii) Determine the eigenvalues of the matrix

$$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$$

and verify that $f\left(\frac{\pi}{8}\right)$ is equal to the larger of them.

(iii) There is a second turning point in $[0, \pi]$. What is it?