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AS1056 - Chapter 12, Tutorial 2. 22-02-2024. Notes.

Exercise 12.10

(i) Take, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; the eigenvalues of A satisfy the characteristic equation of A:

$$\det(A - \lambda I) = \left| \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \right| = (a-\lambda)(d-\lambda)) - bc = 0$$

operating,

$$ad - \lambda - \lambda d + \lambda^2 - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

We want to choose a, b, c and d in such a way that the solutions of this equation are $\lambda = 3$ and $\lambda = -1$. That is, remembering that equations can be expressed in factored form,

$$\lambda^2 - {\color{red}(a+d)}\lambda + {\color{blue}ad - bc} = \underbrace{\underbrace{(\lambda-3)(\lambda+1)}_{= \lambda^2 + \lambda - 3\lambda - 3}}_{= \lambda^2 - {\color{red}2}\lambda {\color{blue}-3}} = 0$$

Therefore, we need $a + d = 2$ and $ad - bc = 3$. For the sake of simplicity, let us just take $a = d = 1$. Then, we have $1 - bc = -3$, i.e., $bc = 4$; and again for the sake of simplicity, we just take $b = c = 2$. And finally we get that a posible solution for A is:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

(ii) Given the characteristic equation of A, $\lambda^2 - 2\lambda - 3 = 0$, we want to show that $A^2 - 2A - 3I = \mathbf{O}$.

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

And,

$$A^2 - 2A - 3I = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Exercise 12.12

$$f(\theta) = \begin{pmatrix}\cos(\theta) & \sin(\theta)\end{pmatrix} \underbrace{\begin{pmatrix}6 & 2\\ 2 & 2\end{pmatrix}}_{A} \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}, \quad \theta \in [0, \pi]$$

Theory suggests:

- $\max_\theta f(\theta) = \max(\lambda_1, \lambda_2), \quad \theta \in [0, \pi]$
- $\min_\theta f(\theta) = \min(\lambda_1, \lambda_2), \quad \theta \in [0, \pi]$

where λ_1 and λ_2 denote the eigenvalues of matrix A.

Recall the following trigonometric identities,

- Pythagorean formula for sines and cosines:

  $\sin^2(\theta) + \cos^2(\theta) = 1$

- Double angle formulas for sine and cosine:

  $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
  $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

(i) We want to show that $\theta = \frac{\pi}{8}$ and that it is a maximum. Let us start by trying to simplify the quadratic form $f(\theta)$:

$$f(\theta) = \begin{pmatrix}6\cos(\theta) + 2\sin(\theta) & 2\cos(\theta) + 2\sin(\theta)\end{pmatrix} \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} =$$

$$6\cos^2(\theta) + 2\sin(\theta)\cos(\theta) + 2\cos(\theta)\sin(\theta) + 2\sin^2(\theta) = 6\cos^2(\theta) + 2\sin^2(\theta) + 4\sin(\theta)\cos(\theta) =$$

$$= 4\cos^2(\theta) + 4\sin^2(\theta) + 2\cos^2(\theta) - 2\sin^2(\theta) + 4\sin(\theta)\cos(\theta) =$$

$$= 4\underbrace{[\cos^2(\theta) + \sin^2(\theta)]}_{=1} + 2\underbrace{[\cos^2(\theta) - \sin^2(\theta)]}_{=\cos(2\theta)} + \underbrace{4\sin(\theta)\cos(\theta)}_{2 \times \underbrace{2\sin(\theta)\cos(\theta)}_{=\sin(2\theta)}}$$

$$= 4 + 2\cos(2\theta) + 2\sin(2\theta)$$

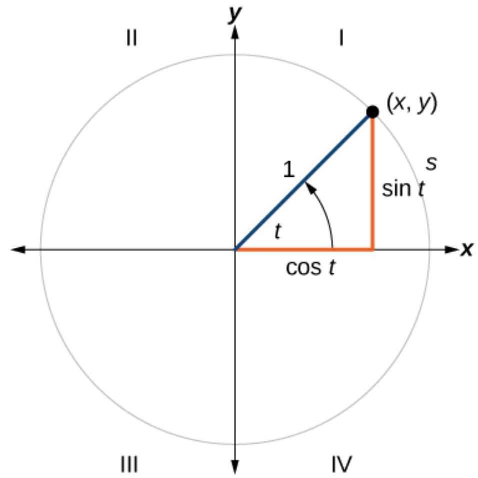
Taking the first derivative and equating to zero:

$$f'(\theta) = -4\sin(2\theta) + 4\cos(2\theta) = 0; \quad \cancel{4}\cos(2\theta) = \cancel{4}\sin(2\theta)$$

That is,

$$\cos(2\theta) = \sin(2\theta) \quad \text{or} \quad \frac{\sin(2\theta)}{\cos(2\theta)} = \tan(2\theta) = 1$$

Now, given the above identities, how much is 2θ? Maybe you remember by heart which angles $\theta \in [0, \pi]$ have a tangent equal to 1. In such case you can immediately deduce which are the turning points we're looking for. If you don't remember, you can note that $\cos(2\theta) = \sin(2\theta)$ implies that 2θ is an angle for which the values of sine and cosine are equal. Having in mind the unit circle,



it can come to us that —within the first period of these trigonometric functions[1]—, the angles at which sine=cosine are 45° (= $\frac{\pi}{4}$ radians) and 225° (= $\frac{5}{4}\pi$ radians; i.e., $45° + 180°$). Due to the periodic nature of sine/cosine functions there's an infinite number of solutions. In fact, the solution is any θ such that

$$2\theta = \frac{\pi}{4} + n\pi, \text{ i.e., } \theta = \frac{\pi}{8} + n\frac{\pi}{2}, \quad n \in \mathbb{Z}$$

Nevertheless, remind that we were told that $\theta \in [0, \pi]$, and thus the two only permissible solutions are indeed $\theta = \frac{\pi}{8}$ and $\theta = \frac{5}{8}\pi$.

Hence, we conclude that $\theta = \frac{\pi}{8}$ is a turning point. Now, let us show it is a maximum. First take the second derivative:

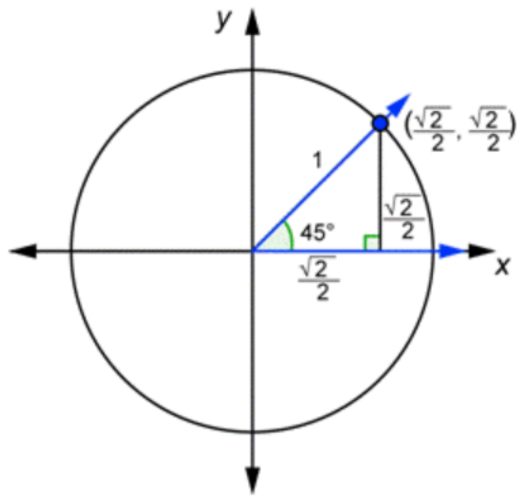
$$f''(\theta) = -8\cos(2\theta) - 8\sin(2\theta) = -8[\cos(2\theta) + \sin(2\theta)]$$

Evaluating the second derivative for $\theta = \frac{\pi}{8}$:

$$f''\left(\frac{\pi}{8}\right) = -8\left[\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right]$$

[1]Period of the sine and cosine is 2π.

If you don't remember how much is $\cos(45°) = \sin(45°)$, consider an isosceles triangle (two equal sides and two equal angles) embedded on the unit circle and try to find it out by just using by Pythagoras theorem what $\cos(45°) = \sin(45°)$ should be. Note that the hypotenuse of such triangle will correspond to the radius of the unit circle:



Therefore,

$$f''\left(\frac{\pi}{8}\right) = -8\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = -8\sqrt{2} < 0$$

Since $\theta = \frac{\pi}{8}$ is a turning point and $f''\left(\frac{\pi}{8}\right) < 0$, we conclude that $\theta = \frac{\pi}{8}$ is a maximum of f.

(ii) To start with, let us work out:

$$f\left(\frac{\pi}{8}\right) = 4 + 2\underbrace{\cos\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}} + 2\underbrace{\sin\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}} = 4 + 2\sqrt{2} = 4 + 2\sqrt{2}$$

On the other hand, the eigenvalues of A satisfy its characteristic equation:

$$\det(A - \lambda I) = \left|\begin{pmatrix} 6-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix}\right| = (6-\lambda)(2-\lambda) - 4 = 12 - 6\lambda - 2\lambda + \lambda^2 - 4 = \lambda^2 - 8\lambda + 8 = 0$$

$$\begin{aligned} \lambda &= \frac{8 \pm \sqrt{8^2 - 4 \times 1 \times 8}}{2 \times 1} \\ &= \begin{cases} \lambda_1 = 4 + \dfrac{\sqrt{32}}{2} = 4 + 2\sqrt{2} \\ \lambda_2 = 4 - \dfrac{\sqrt{32}}{2} = 4 - 2\sqrt{2} \end{cases} \end{aligned}$$

Thus, we confirm that $\max_{\theta \in [0,\pi]} f(\theta) = f(\frac{\pi}{8}) = \max(\lambda_1, \lambda_2) = \lambda_1 = 4 + 2\sqrt{2}$.

(iii) Just check whether $\theta = \frac{5}{8}\pi$ is a minimum and whether $f(\frac{5}{8}\pi) = \min(\lambda_1, \lambda_2) = \lambda_2 = 4 - 2\sqrt{2}$.