AS1056 - Mathematics for Actuarial Science. Chapter 10, Tutorial 2.

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Refreshing some concepts: Geometric interpretation of vectors and matrices

In linear algebra, we are interested in equations of the following form:

$$\mathbf{u} = A\mathbf{v}$$

where $\mathbf{u} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, and $\mathbf{v} \in \mathbb{R}^n$.

- One way to think about this equation is that A represents a system of m linear equations, each with n variables, and $\mathbf{v}$ represents the unknowns to this system.
- But there is another way to think of the matrix A, which is as a linear function f from R^n to R^m,

  $$f(\mathbf{v}) = A\mathbf{v}$$

  i.e. as a mapping $A : R^n \to R^m$.

- If $m = n$, meaning the matrix A is square, then the linear transformation maps $\mathbb{R}^m$ onto itself. This can represent various geometric transformations (rotations, reflections, scaling, etc.) within the same dimensional space, $\mathbb{R}^m$.
- The *unit square* plays a significant role in the geometric interpretation of matrices as linear transformations, serving as a fundamental and intuitive example for visualising how transformations affect space.

With a geometrical understanding of matrices as linear transformations, many concepts in linear algebra are easier to appreciate: invertibility, rank, eigenvalues,...

Determinant of 2×2 Matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then det(A) $= ad - bc$.

In geometrical terms:

- A square matrix represents a mapping of a space on to itself, $A : \mathbb{R}^2 \to \mathbb{R}^2$.
- The *absolute value of the determinant* is the area of the image of the unit square under the mapping A.

Take $A \in \mathbb{R}^{2\times 2}$ (i.e., a 2×2 matrix), then:

- If det(A) > 0, the linear transformation **preserves** the orientation of the vertices of the shape it transforms.
- If det(A) < 0, the linear transformation **reverses** the orientation of the vertices of the shape it transforms.
  $\longrightarrow$If you start at one corner of a unit square and move around its edges in an *clockwise* direction, after the transformation by a matrix with a negative determinant, the corresponding path around the transformed figure (i.e., the image of the original square) would be in a *anticlockwise* direction
- If det(A) $= 0$, then matrix A represents a linear transformation that collapses the space to a lower dimension (such as a square becoming a line or a 3D object becoming a flat shape).

The determinant gives the **scaling factor** (in terms of the square area) and the **orientation** induced by the mapping represented by A.

Exercise 11.7

(i) For each of the following matrices, draw the unit square and its image under the mapping which the matrix represents.

(ii) In each of the three cases, apply the mapping a second time. What would you expect to happen if the mapping were applied repeatedly?

a) $$A_1 = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

Exercise 11.9

(i) What is the area of the image of the unit square under the mapping represented by

$$A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}?$$

(ii) Write down matrices representing

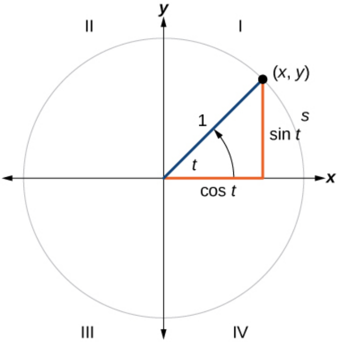
a) An enlargement by which areas are multiplied by a factor of 3.

(ii) Write down matrices representing

b) A rotation through an angle of $\pi/6$.

Hint: A rotation through then angle θ is represented by:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



(ii) Write down matrices representing

   c) A rotation through an angle of $\pi/6$ and an enlargement which multiplies areas by 3.

Exercise 11.11

A is a 2×2 matrix with determinant equal to 0.

(i) Assume that the first row of A is not equal to $\begin{pmatrix} 0 & 0 \end{pmatrix}$. Show that the second row of A is a multiple of the first row.

(ii) Assume that the first column of A is not equal to $\begin{pmatrix} 0 & 0 \end{pmatrix}^{\mathbf{T}}$. Show that the second columns of A is a multiple of the first.