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Exercise 10.10

- Two(n): "n is equal to two".
- Even(n): "n is equal to an even number".
- Prime(n): "n is equal to a prime number".

"If n is a prime number and $n \neq 2$, then n is odd". In logical notation:

$\forall n \ [[\text{Prime}(n) \land \neg\text{Two}(n)] \implies \neg\text{Even}(n)]$

The contrapositive of this statement is:

$\forall n \ [\text{Even}(n) \implies \neg[\text{Prime}(n) \land \neg\text{Two}(n)] = [\neg\text{Prime}(n) \lor \text{Two}(n)]]$

Putting this in words: "If n is ane even number, then either n is not prime or n is equal to two". Indeed, the only even prime number is 2.

Exercise 10.7

W.t.s. that $\forall n \in \mathbb{Z}, \exists \ k \in \mathbb{Z}$ s.t. $n^2 + n = 2k$. Let us recall that, by definition, an even number is an integer of the form $n = 2k$, $k \in \mathbb{Z}$. Therefore, given the definition of even number, to prove the latter statement we only need to show that $n^2 + n$ is even.

$n \in \mathbb{Z}$, we don't know if even or odd. To show that $n^2 + n$ is even let us consider the case where n is even and the case where n is odd, and try to say something about the evenness/oddness of $n^2 + n$

1. Assume n is even and consider the following predicates,

   - P: Even(n)
   - Q: $\forall n \ [\text{Even}(n) \implies \text{Even}(n^2)]$
     Proof: If n is even, then by definition of even number, there exists $k \in \mathbb{Z}$ s.t. $n = 2k$; thus, $n^2 = 4k^2 = 2\underbrace{(2k^2)}_{k'} = 2k'$, and therefore, by definition, n^2 is also even.
   - R: $\forall n, \forall m \ [\text{Even}(n) \land \text{Even}(m) \implies \text{Even}(n+m)]$
     Proof: Also using the definition of even number, take,

     $$\left.\begin{array}{r} n = 2k \\ m = 2k' \end{array}\right\} \text{then, } n + m = 2k + 2k' = 2(k+k') = 2k''\text{; thus, } n+m \text{ is also even.}$$

   Therefore, $\dfrac{P; Q; R}{\therefore \text{Even}(n^2+n)}$.

2. Assume n is odd and consider the following predicates,

   - P': $\neg\text{Even}(n)$
   - Q': $\forall n\,[\neg\text{Even}(n) \implies \neg\text{Even}(n^2)]$
     Proof: By complementarity of statement Q. You can also use the definition of odd number.
   - R': $\forall n, \forall m[\neg\text{Even}(n) \wedge \neg\text{Even}(m) \implies \text{Even}(n+m)]$
     Proof: Using the definition of odd number, take,

     $$\left.\begin{array}{l} n = 2k+1 \\ m = 2k'+1 \end{array}\right\} \text{then, } n+m = 2(k+k')+2 = 2\underbrace{(k+k'+1)}_{k''} = 2k''; \; n+m \text{ is even.}$$

   Therefore, $\dfrac{P'; Q'; R'}{\therefore\ \text{Even}(n^2+n)}$.

We conclude that n^2+n is always even. And then, by the definition of even number, $\exists\, k \in \mathbb{Z}$ s.t. $n^2+n = 2k$.

Exercise 10.12

The proof goes wrong in point 5, by making the implicit assumption that the two groups of n people —Group A={Person 1, ..., Person n} and Group B={Person 2, ..., Person $n+1$}— to which the induction hypothesis is applied have a common element. This is not true for $n = 1$, in which case there is no overlap between the groups (i.e., Group A $\cap$ Group B $= \emptyset$). The inductive step presented would indeed have worked if we assume $n \geq 2$ (i.e., if $n \geq 2$, $S(n) \implies S(n+1)$); however, in such circumstances it would be the base case the one failing in this proof by mathematical induction.