AS1056 - Mathematics for Actuarial Science. Chapter 1, Tutorial 2.

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Exercise 1.4

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if for any $\varepsilon > 0$, we can find some $\delta = \delta(\varepsilon) > 0$ such that whenever $x > \delta$ we have that $L - \varepsilon < f(x) < L + \varepsilon$, i.e., $|f(x) - L| < \varepsilon$

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So, we want to show,

$$\frac{2x^3 - 1}{(x+1)^3} \xrightarrow{x\to\infty} 2, \text{ i.e., } \left| \frac{2x^3 - 1}{(x+1)^3} - 2 \right| < \varepsilon \text{ for } x > \delta(\varepsilon);\ \delta > 0,\ \varepsilon > 0$$

Of course you can always calculate the limit with your favourite technique:

$$\lim_{x\to\infty} \frac{2x^3 - 1}{(x+1)^3} = \lim_{x\to\infty} \frac{2x^3 - 1}{x^3 + 3x^2 + 3x + 1} \underset{\uparrow}{=}$$

divide up/down by x^3

$$= \lim_{x\to\infty} \frac{2 - \overbrace{\frac{1}{x^3}}^{\to 0}}{1 + \underbrace{\frac{3}{x}}_{\to 0} + \underbrace{\frac{3}{x^2}}_{\to 0} + \underbrace{\frac{1}{x^3}}_{\to 0}} = 2$$

Or using l'Hôpital's rule that tells you that if:

1. $f(x)$, $g(x)$ differentiable
2. $\frac{d}{dx}g(x) \neq 0$
3. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$

then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}g(x)} = L$$

Where c and L are any real number or $\pm\infty$.

$$\lim_{x\to\infty} \frac{2x^3 - 1}{(x+1)^3} = \lim_{x\to\infty} \frac{6x^2}{3(x+1)^2} = \lim_{x\to\infty} \frac{12x}{6(x+1)} = \lim_{x\to\infty} \frac{12}{6} = 2$$

But calculating the limit with any of this techniques wouldn't be a foundational rigorous proof!! Indeed on the first calculation you're relying in different limit properties and on the second calculation you're relying on l'Hôpital's rule. And you should prove this things for the proof to be complete (which might be quite complicated...)

Exercise 1.6

Suppose $x = \cos\left(\frac{1}{2}t\right)$ and $y = 2\sin(t)$ for $t \in [0, \pi]$.

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Trigonometric identities

- $2\cos^2(\theta) - 1 = \cos(2\theta)$ (double angle formulas for sine and cosine)
- $\cos^2(\theta) + sin^2(\theta) = 1$ (Pythagorean formula for sines and cosines)

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- $t \in [0, \pi]$
  - $x = \cos\left(\frac{1}{2}t\right)$
    - if $t = 0 \rightarrow x = \cos(0^\circ) = 1$
    - if $t = \pi \rightarrow x = \cos(90^\circ) = 0$

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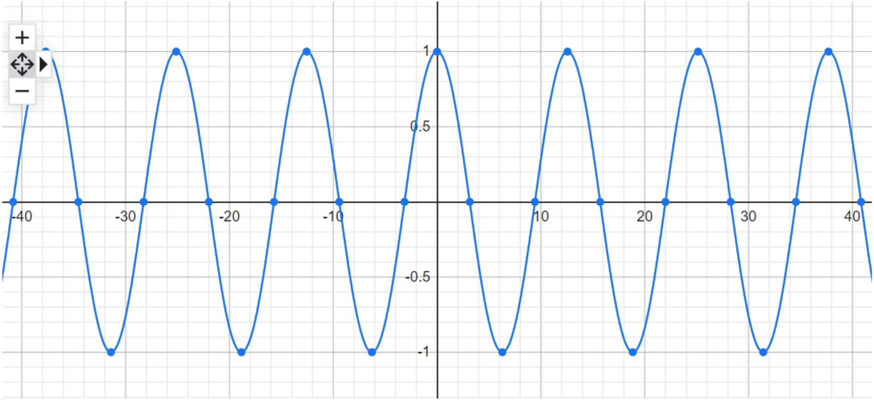
  - $y = 2\sin(t)$
    - if $t = 0 \rightarrow y = 2 \times \sin(0°) = 0$
    - if $t = \pi \rightarrow y = 2 \times \sin(180°) = 0$

Recall to check whether your calculator is radians or degrees (if in radians you'd put π if in degrees you'd put $180°$)

Graph for $\cos(0.5t)$



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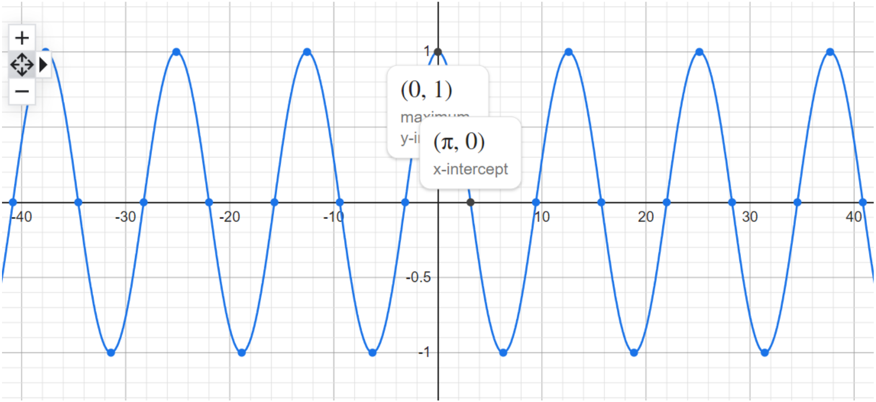
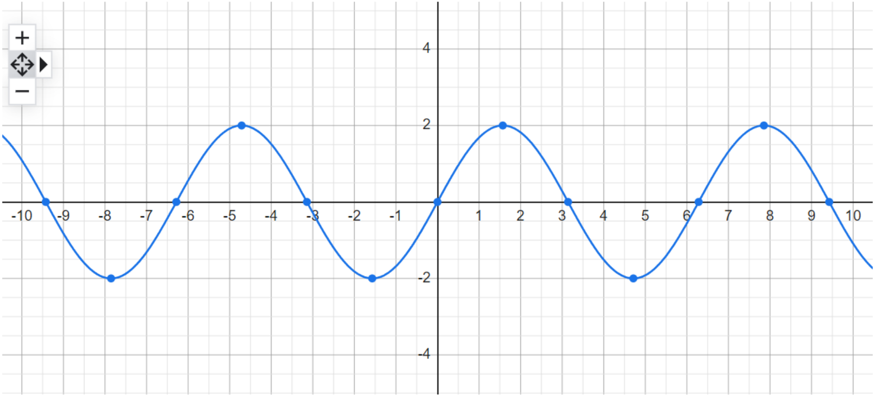
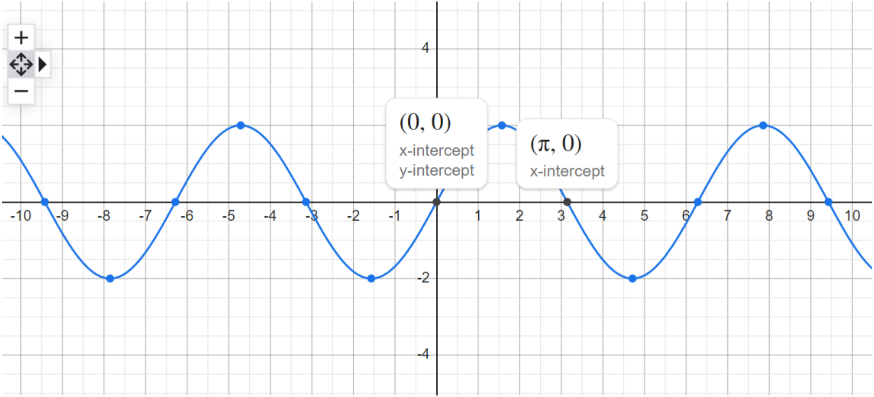


Graph for $2\sin(t)$



Graph for $2\sin(t)$



(iii) Work out the inverse function g such that $x = g(y)$.

Hint:

Consider the previously obtained expression $4x^4 - 4x^2 + \frac{1}{4}x^4 = 0$