Emilio Luis Sáenz Guillén
PhD in Actuarial Science Student

# AS1056 - Chapter 0, Tutorial 2. 13-10-2023. Notes.

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Hello everybody, sorry I think I was not completely clear with some of the exercises I solved today, so let me make some clarifications as follows:

**Exercise B.5 (ii)**

If you take the infinitely many rational numbers away from the infinitely many real numbers, are you left with: (c) an uncountably infinite set.

I will show that this is correct by showing that $\mathbb{R}$ is countable **if and only if** $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$ is countable, in other words: $\mathbb{R}$ countable $\iff$ $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$ countable. Note that if the latter is correct, it implies then that if $\mathbb{R}$ is uncountable (which it is actually the case) then, $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$ is uncountable (of course, I am proving the latter because it is easier).

*Proof:*

$(\Longrightarrow)$ : Assume that $\mathbb{R}$ is countable. Now, note that $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q} \subseteq \mathbb{R}$ which implies that $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$ is countable since it is contained within a countable set, $\mathbb{R}$ (since we assumed $\mathbb{R}$ to be countable).

$(\Longleftarrow)$ : Assume now that $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$ is countable. Consider the set $\mathbb{R}\backslash\mathbb{Q} \cup \mathbb{Q}$, which is countable since the rational numbers set $\mathbb{Q}$ is countable (check your lecture notes) and we have assumed $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$ to be countable (and as we have shown on B.5(i) the union of countable sets is countable). Now note that $\mathbb{R} \subseteq \mathbb{R}\backslash\mathbb{Q} \cup \mathbb{Q}$ (indeed these two sets are equal!); therefore, $\mathbb{R}$ is countable since it is contained within a countable set.

ALTERNATIVE ARGUMENT:

We know that $\mathbb{R}$ (the set of real numbers) is uncountably infinite. We know that $\mathbb{Q}$ (the set of rational numbers) is countably infinite. Assuming for the sake of contradiction that $\mathbb{I} = \mathbb{R}\backslash\mathbb{Q}$ is countably infinite, then the union of two countably infinite sets, namely $\mathbb{R}\backslash\mathbb{Q}$ and $\mathbb{Q}$, would also be countably infinite (as the union of two countable sets is countable). However, $\mathbb{R}\backslash\mathbb{Q} \cup \mathbb{Q} = \mathbb{R}$ and we know $\mathbb{R}$ is uncountably infinite, leading to a contradiction. Thus, $\mathbb{R}\backslash\mathbb{Q}$ must also be uncountably infinite.

**Exercise E.2 (i)** Show that 6 makes exactly 3 appearances, i.e., that it cannot occur again lower down in the triangle.

We can find out (by hand, let's say) which are the 3 appearances of 6 in Pascal's triangle: $\binom{6}{1}$, $\binom{4}{2}$ and $\binom{6}{5}$. Now, what might be more challenging is to show that 6 will not appear again in Pascal's triangle.

Note that a specific integer can appear at most once in each of the sets $\left\{{}^{n}C_1 : n = 1, 2, 3, ...\right\}$, $\left\{{}^{n}C_2 : n = 1, 2, 3, ...\right\}$,... This is because the number in the position ${}^{n}C_k$ (for any $k$ and any $n$) is derived from the sum of two numbers above it in the triangle, and hence these grow as you move down the triangle, that is: $...{}^{n-1}C_k < {}^{n}C_k < {}^{n+1}C_k....$

- Consider the set $\left\{{}^{n}C_1 : n = 1, 2, 3, ...\right\}$, can we find an $n$ such that ${}^{n}C_1 = 6$? Yes, as we have already mentioned, this is 6 since $\binom{6}{1} = 6$.
- Consider the set $\left\{{}^{n}C_2 : n = 1, 2, 3, ...\right\}$, can we find an $n$ such that ${}^{n}C_2 = 6$? Yes, as we have already mentioned, is 4 since $\binom{4}{2} = 6$.

- Consider the set $\left\{{}^{n}C_3 : n = 1, 2, 3, ...\right\}$, can we find an $n$ such that ${}^{n}C_3 = 6$? No since: $\binom{3}{3} = 1$, $\binom{4}{3} = 4$, $\binom{5}{3} = 10$,... and any other $\binom{k}{3}$ for $k > 5$ will be $> 10$ as we have discussed before.
- Consider the set $\left\{{}^{n}C_4 : n = 1, 2, 3, ...\right\}$, can we find an $n$ such that ${}^{n}C_4 = 6$? No since: $\binom{4}{4} = 1$, $\binom{5}{4} = 5$, $\binom{6}{4} = 15$,... and any other $\binom{k}{4}$ for $k > 6$ will be $> 15$ as we have discussed before.
- Consider the set $\left\{{}^{n}C_5 : n = 1, 2, 3, ...\right\}$, can we find an $n$ such that ${}^{n}C_5 = 6$? Yes, as we have already mentioned, this is 6 since $\binom{6}{5} = 6$.
- Consider the set $\left\{{}^{n}C_6 : n = 1, 2, 3, ...\right\}$, can we find an $n$ such that ${}^{n}C_6 = 6$? No since: $\binom{6}{6} = 1$, $\binom{7}{6} = 7$, $\binom{8}{6} = 28$,... and any other $\binom{k}{6}$ for $k > 8$ will be $> 28$ as we have discussed before.

And so on...